Sharp spectral asymptotics for four-dimensional Schrödinger operator with a strong degenerating magnetic field.

Victor Ivrii *

2nd February 2008

Abstract

I continue analysis of the Schrödinger operator with the strong degenerating magnetic field, started in [Ivr6]. Now I consider 4-dimensional case, assuming that magnetic field is generic degenerated and under certain conditions I derive spectral asymptotics with the principal part $\asymp h^{-4}$ and the remainder estimate $O(\mu^{-1/2}h^{-3})$ where $\mu \gg 1$ is the intensity of the field and $h \ll 1$ is the Plank constant; $\mu h \le 1$.

These asymptotics can contain correction terms of magnitude $\mu^{5/4}h^{-3/2}$ corresponding to the short periodic trajectories.

0 Introduction

0.0 Preface

I continue analysis of he Schrödinger operator with the strong degenerating magnetic field, started in [Ivr6]

$$A = \frac{1}{2}\Bigl(\sum_{j,k} P_j g^{jk}(x) P_k - V\Bigr), \qquad P_j = D_j - \mu V_j \tag{0.1}$$

where g^{jk}, V_j, V are smooth real-valued functions of $x \in \mathbb{R}^2$ and (g^{jk}) is positive-definite matrix, $0 < h \ll 1$ is a Planck parameter and $\mu \gg 1$ is a coupling parameter. I assume that A is a self-adjoint operator.

*Work was partially supported by NSERC grant OGP0138277.

Now I consider 4-dimensional case, assuming that magnetic field is generic degenerated, i.e. of Martinet-Roussarie type [Ma, Rou] which will be described in details in subsection 1.1. Here I just mention that in this case magnetic field F (which currently is considered as a closed 2-form) degenerates on the manifold Σ of dimension 3 and $\dim \operatorname{Ker} F = 2$ at Σ. Further, $\dim(\operatorname{Ker} F \cap T\Sigma) = 1$ at $\Sigma \setminus \Lambda$ and $\operatorname{Ker} F \subset T\Sigma$ at Λ where $\Lambda \subset \Sigma$ is 1-dimensional manifold; furthermore, an angle between $\operatorname{Ker} F$ and $T\Sigma$ is exactly of magnitude $\operatorname{dist}(x, \Lambda)$. Finally, in an appropriate coordinates $\Sigma = \{x_1 = 0\}$, $\Lambda = \{x_1 = x_3 = x_4 = 0\}$ and *magnetic lines* $\frac{dx}{dt} \in \operatorname{Ker} F \cap T\Sigma$ are circles $\{x_1 = 0, x_2 = \mathsf{const}, x_3^2 + x_4^2 = \mathsf{const}\}$.

My goal is to find asymptotics of $\int e(x, x, 0)\psi(x)\,dx$ with respect to h, μ, where $e(x, y, \tau)$ is the Schwartz kernel of the spectral projector of A and ψ is a smooth function supported in the vicinity of Σ. I assume that $\mu h \le \mathsf{const}$ (otherwise $e(x, y, 0) = O(\mu^{-\infty})$).

In the next paper I am gong to analyze the case of the generic non-degenerated field.

0.1 Assumptions, notations and results

So, magnetic field is characterized by $F_{jk} = \partial_j V_k - \partial_k V_j$.

However, from the point of view of the classical and quantum dynamics and spectral asymptotics properties of $(F^j_k) = (g^{jl})(F_{lk})$ are more important than those of (F_{jk}). Let $\pm i f_1, \pm i f_2$ be eigenvalues of (F^j_k), $f_j \ge 0$. Then with the correct choice of notations $f_1 \asymp \operatorname{dist}(x, \Sigma)$ and $f_2 \asymp 1$.

My first statement holds almost without any further assumptions:

Theorem 0.1. *Let F be of Martinet-Roussarie type and*

$$V \ge \epsilon_0 > 0. \tag{0.2}$$

Let ψ be supported in the small vicinity of Σ. Then

$$|\int \bigl(e(x, x, 0) - \mathcal{E}^{\mathsf{MW}}(x, 0)\bigr)\psi(x)\,dx| \le C\mu^{-1/2}h^{-3} + C\mu^2 h^{-2} \tag{0.3}$$

where

$$\mathcal{E}^{\mathsf{MW}}(x, \tau) = (2\pi)^{-2}\mu^2 h^{-2} \sum_{(m,n)\in\mathbb{Z}^{+\,2}} \theta\bigl(2\tau + V - (2m+1)\mu h f_1 - (2n+1)\mu h f_2\bigr) f_1 f_2 \sqrt{g} \tag{0.4}$$

is Magnetic Weyl Expression (see [Ivr3] for the general case), $g = \det(g^{jk})^{-1}$.

Remark 0.2. Without any additional conditions remainder estimate (0.3) is sharp. An extra term $O(\mu^2h^{-2})$ in its right-hand expression is due to the possibility that for some pair $(m,n)\in\mathbb{Z}^{+\,2}$ function $V-(2m+1)\mu hf_1-(2n+1)\mu hf_2$ identically vanishes in the vicinity of some point of Σ. However it is not as bad as it could be in the case of constant comeasurable f_1,f_2 in which case $V-(2m+1)\mu hf_1-(2n+1)\mu hf_2$ could vanish up to $\asymp(\mu h)^{-1}$ pairs $(m,n)\in\mathbb{Z}^{+\,2}$ and the remainder estimate would be the worst possible $O(\mu h^{-3})$.

In the next few statements I am improving remainder estimate (0.3).

Theorem 0.3. *Let F be of Martinet-Roussarie type, conditions (0.2) and*

$$|(\frac{V}{f_2})-(2n+1)\mu h|+|\nabla_\Sigma(\frac{V}{f_2})|\le\epsilon_0\implies|\det\mathsf{Hess}_\Sigma(\frac{V}{f_2})|\ge\epsilon_0\qquad\forall n\in\mathbb{Z}^+\tag{0.5}$$

be fulfilled. Let ψ be supported in the small vicinity of Σ. Then

$$|\int\bigl(e(x,x,0)-\mathcal{E}^{\mathsf{MW}}(x,0)\bigr)\psi(x)\,dx-\int_\Sigma\mathcal{E}^{\mathsf{MW}}_{\mathsf{corr}}(x',\tau)\psi(x')\,dx'|\le C\mu^{-1/2}h^{-3}\tag{0.6}$$

where dx' is the standard density on Σ and $\mathcal{E}^{\mathsf{MW}}_{\mathsf{corr}}(x',\tau)$ is a correction term (see (4.51) which is $O(\mu^{5/4}h^{-3/2})$.

Remark 0.4. (i) Correction terms $\mathcal{E}^{\mathsf{MW}}_{\mathsf{corr}}$ and $\mathcal{E}^{\mathsf{MW}}_{\mathsf{corr}}$ are of completely different nature;
(ii) The first term in the left-hand expression of (0.5) is meaningful only in the case of the *very strong magnetic field* $\mu h\ge\epsilon$; so as $\mu h\le\epsilon$ condition (0.5) is equivalent to

$$|\nabla_\Sigma(\frac{V}{f_2})|\le\epsilon_0\implies|\det\mathsf{Hess}_\Sigma(\frac{V}{f_2})|\ge\epsilon_0;\tag{0.7}$$

(iii) One can weaken condition (0.5) to

$$(0.8)_q\quad\exists n\in\mathbb{Z}^+:\ |\frac{V}{f_2}-(2n+1)f_2\mu h|+|\nabla_\Sigma(\frac{V}{f_2})|\le\epsilon_0\implies$$
$$\mathsf{Hess}_\Sigma(\frac{V}{f_2})\text{ has at least }q\text{ eigenvalues with absolute values greater than }\epsilon_0$$

with $q=1,2,3$ depending on the magnitude of μ; as $\mu h\le\epsilon$ it is equivalent to

$$(0.9)_q\quad|\nabla_\Sigma(\frac{V}{f_2})|\le\epsilon_0\implies$$
$$\mathsf{Hess}_\Sigma(\frac{V}{f_2})\text{ has at least }q\text{ eigenvalues with absolute values greater than }\epsilon_0;$$

obviously (0.5), (0.7) are equivalent to $(0.8)_3$, $(0.9)_3$ respectively;
(iv) Under stronger condition

$$|(\frac{V}{f_2}) - (2n+1)\mu h| + |\nabla_\Sigma(\frac{V}{f_2})| \ge \epsilon_0 \qquad \forall n\in\mathbb{Z}^+ \tag{0.10}$$

the proof of theorem 0.3 is much easier; as $\mu h \ll 1$ this condition is equivalent to

$$|\nabla_\Sigma(\frac{V}{f_2})| \ge \epsilon_0. \tag{0.11}$$

(v) In the very special cases (see Appendix A) one can derive remainder estimate (0.6) without condition (0.5) or even $(0.8)_1$. Then the correction term could be larger, up to $O(\mu^{1/4}h^{-5/2})$.

I will also prove

Theorem 0.5. *Let F is of Martinet-Roussarie type and condition (0.10) be fulfilled. Then as ψ is supported in the small vicinity of $\Sigma\cap\{V=0\}$ asymptotics (0.6) holds with $\mathcal{E}^{\mathsf{MW}}_{\mathsf{corr}}(x',\tau)=0$.*

0.2 Plan of the paper

First of all, in section 1 I study the weak magnetic case $\mu\le h^{-\delta}$ with sufficiently small exponent $\delta>0$ and derive remainder estimate $O(\mu^{-1/2}h^{-3})$ as the main part is given by the standard Weyl formula. I also analyze there the geometry of the degenerate magnetic field and the corresponding classical dynamics.

Then, assuming that magnetic field is not weak, $h^{-\delta}\le\mu\le ch^{-1}$ in section 2 I consider the different canonical forms of the operator in question; this canonical forms contain powers of μ^{-1}. There are universal canonical forms and also specific canonical forms as $\mathsf{dist}(x,\Sigma)\gg\mu^{-1/2}$ or as $\mathsf{dist}(x,\Lambda)\gg\mu^{-1/2}$.

Then in section 3 I derive remainder estimates $O(\mu^{-1/2}h^{-3}+\mu^2h^{-2})$ in the general case and $O(\mu^{-1/2}h^{-3})$ as $\mu\ge h^{-2/5}$ and some non-degeneracy condition (depending on the magnitude of μ) is fulfilled. The main part of this estimate is given by the *standard intermediate formula*

$$h^{-1}\int_{-\infty}^{0}\Bigl(F_{t\to h^{-1}\tau}\bar\chi_T(t)\Gamma uQ_y^t\Bigr)\,d\tau \tag{0.12}$$

where u is the Schwartz kernel of $e^{ih^{-1}tA}$, $Q=I$ and $\bar\chi_T(t)$ equals 1 as $|t|\le\frac{1}{2}T$ and vanishes as $|t|\ge T$, T is rather arbitrary from interval $[T_0,T_1]$ and the main fight is to

make T_0 as small and T_1 as large as possible. More precisely, the main part of asymptotics is given by the sum of expressions (0.12) with $Q = Q_{(\iota)}$ making partition of unity and $T = T_\iota \in [T_{(\iota)0}, T_{(\iota)1}]$.

After this in section 4 I calculate (0.12) in more explicit way.

Finally, Appendices A are devoted to the analysis of some special cases.

1 Weak magnetic field

1.1 Geometry of degenerating magnetic field

So let $F_{jk}(x) = \partial_j V_k - \partial_k V_j$ be components of the matrix intensity of magnetic field; then $\omega = \sum_{j,k} F_{jk} dx_j \wedge dx_k = d(\sum_k V_k dx_k)$ is the corresponding magnetic 2-form. According to [Ma], the generic form in dimension 4 never degenerates completely and has rank 2 on the submanifold Σ of codimension 1; more precisely, let

$$\Sigma \overset{\mathsf{def}}{=} \bigl\{x : \operatorname{rank} \omega(x) = 2\bigr\}; \tag{1.1}$$

then

(1.2) For a generic 2-form ω Σ is the smooth manifold of codimension 1 and if $\pm i f_1$, $\pm i f_2$ are eigenvalues of the corresponding matrix (F_{jk}), $f_2 \ge f_1 \ge 0$ [1)] then $f_2(x) \ge \epsilon$ and $f_1(x) \asymp \operatorname{dist}(x, \Sigma)$.

Let us consider $\operatorname{Ker} F(x) \cap T_x\Sigma$ in points of Σ. At each point it can be of dimension 1 or 2; let

$$\Lambda \overset{\mathsf{def}}{=} \bigl\{x \in \Sigma : \dim\bigl(\operatorname{Ker} F(x) \cap T_x\Sigma\bigr) = 2\bigr\}; \tag{1.3}$$

according to [Rou]

(1.4) For a generic 2-form ω Λ is the smooth manifold of dimension 1; moreover in the appropriate coordinates

$$\omega = dx_1 \wedge dx_2 + x_3 dx_2 \wedge dx_3 + d\bigl(x_1x_3 + x_2x_4 - \frac{1}{3}x_3^3\bigr) \wedge dx_4. \tag{1.5}$$

---

1) This is a temporary notation, corresponding to Euclidean metrics g^{jk}.

One can rewrite (1.5) as

$$d\Bigl(\bigl(x_1-\frac{1}{2}(x_3^2+x_4^2)\bigr)d(x_2+\frac{1}{2}x_3x_4)+(x_3^2+x_4^2)x_1-\frac{1}{4}(x_3^2+x_4^2)^2\Bigr);$$

after transformation $x_2\mapsto x_2-\frac{1}{2}x_3x_4$ I get instead

$$\begin{gathered}\omega = dx_1\wedge dx_2 - x_4dx_1\wedge dx_3 + x_3dx_1\wedge dx_4 + x_3dx_2\wedge dx_3 + x_4dx_2\wedge dx_4+\\ 2\bigl(x_1-\frac{1}{2}(x_3^2+x_4^2)\bigr)dx_3\wedge dx_4 =\\ d\bigl(x_1-\frac{1}{2}r^2\bigr)\wedge dx_2 + r^2dx_1\wedge d\theta + 2\bigl(x_1-\frac{1}{2}r^2\bigr)rdr\wedge d\theta = d\Bigl((x_1-\frac{1}{2}r^2)dx_2+(x_1-\frac{1}{4}r^2)r^2d\theta\Bigr)\end{gathered}\tag{1.6}$$

where $x_3=r\cos\theta$, $x_4=r\sin\theta$. In contrast to the original Roussarie 2-form (1.5) this latter form is obviously (x_3,x_4)-circular symmetric.

Then

$$\Sigma=\{x_1=0\},\quad \Lambda=\{x_1=x_3=x_4=0\}\tag{1.7}$$

and *magnetic lines* defined by

$$\frac{dx}{dt}\in \operatorname{Ker} F(x)\cap T_x\Sigma,\quad x\in\Sigma\tag{1.8}$$

are helices $\{x_1=0, x_3=r\cos\theta, x_4=r\sin\theta, x_2=\mathsf{const}-r^2\theta/2\}$ (with $r=\mathsf{const}$) winging around Λ.

Further, away from Λ one can rewrite

$$\omega = 2x_1dx_1\wedge d\theta + dy\wedge dz,\qquad y=x_1-\frac{1}{2}r^2,\quad z=x_2-2y\theta\tag{1.9}$$

and therefore there I have just a direct sum of two 2-dimensional magnetic fields; the first one is a generic degenerating field and second one is nondegenerate.

1.2 Classical dynamics

Classical dynamics described by the Hamiltonian

$$a(x,\xi)=\frac{1}{2}\Bigl(\sum_{j,k}g^{jk}(x)p_jp_k-V\Bigr),\qquad p_j=\xi_j-\mu V_j\tag{1.10}$$

corresponding to (0.1), is much more complicated than the geometry because it also depends on the metrics (g^{jk}) and the scalar potential V; however it appears that it depends mainly on f_1, f_2 which are eigenvalues of $-i(F^j_k)$ [2)] (where $(F^j_k) = (g^{jl})(F_{lk})$) and V and even on their ratios rather than these functions themselves. It also depends on the partition of energy (see below).

1.2.1 Let us start from considering few examples.

Example 1.1. As F_{jk}, g^{jk} and V are constant (which is not the case in our article) $\mathbb{R}^4 = \mathbb{K}_1 \oplus \mathbb{K}_2$ where $\mathbb{K}_j$ here and below are eigenspaces of $-(F^j_k)^2$ corresponding to eigenvalues f_j^2, $\dim \mathbb{K}_j = 2$. Then the kinetic part of the Hamiltonian splits into $a^0 = a^0_1 + a^0_2$ where a^0_j are Hamiltonians on $\mathbb{K}_j$ and a^0_j are movement integrals and the classical dynamics splits into two *cyclotron* movements along elliptical orbits in $\mathbb{K}_j$ with the angular velocities μf_j; the sizes of these ellipses are $\asymp (\mu f_j)^{-1}(a^0_j)^{1/2}$.

Example 1.2. (i) Let in the frames of the previous example V be variable. Then a_j are no more movement integrals but they evolve with the average speeds[3)] $O(\mu^{-1})$ and in addition to the fast cyclotron movements there appear also slow *drift* movements along $\mathbb{K}_j$ with the speeds of $\asymp \mu^{-1}a^0_j|\nabla_{\mathbb{K}_j}(V/f_j)|$; these drifts are orthogonal to $\nabla_{\mathbb{K}_j}(V/f_j)$. So these drifts depend on the instant energy partition.

(ii) As (g^{jk}) and (F_{jk}) and thus f_j become variable, the spaces $\mathbb{K}_j$ rotate with the average speeds $O(\mu^{-1})$ but the previous statements remain true.

Example 1.3. At points disjoint from Λ one can assume that magnetic field corresponds to $\omega = x_1 dx_1 \wedge dx_2 + dx_3 \wedge dx_4$; I used 2-form (1.9) and redefined coordinates $x' = (x_2, x_3, x_4)$ in the obvious way. Assume that the metrics in these coordinates is Euclidean $g^{jk} = \delta_{jk}$ and that $V = V(x_1, x_2)$. Then $a_1 = a^0_1 - \frac{1}{2}V$ and $a_2 = a^0_2$ are movement integrals as in example 1.1 and the movement splits into two movements again: one of them is the cyclotron movement in (x_3, x_4) and another one is described by Hamiltonian a_1 in coordinates (x_1, x_2). It was studied in details in [Ivr6].

Then there are *outer* and *inner* zones $\mathcal{Z}_{\mathsf{out}}$ and $\mathcal{Z}_{\mathsf{inn}}$ respectively; in the outer zone x_1 should be much larger than the cyclotron radius (associated with a_1) which is $O(\mu^{-1}|x_1|^{-1})$ and this is the case as $\mathcal{Z}_{\mathsf{out}} = \{|x_1| \ge C\mu^{-1/2}\}$. However now a^0_1 is not necessarily disjoint from 0 (which was an assumption in [Ivr6]) and I will need to take it into account and redefine zones.

[2)] In contrast to footnote 1) this is the final definition.

[3)] While the speed of spatial movement is $\asymp 1$ the drift is much slower and one can calculate all functions of x in the instant cyclotron centers.

Still I can conclude from [Ivr6] that if a trajectory $(x(t),\xi(t))$ on an energy level $\le c$ starts from the point $(x(0),\xi(0))$ with $x(0)\in B(0,\frac{1}{2})\cap\{x:|x_1|=\gamma\ge C^2\mu^{-1/2}\}$, then $x(t)\in B(0,\frac{1}{2})\cap\{x:C^{-1}\gamma\le|x_1|\le C\gamma\}$ as $|t|\le\epsilon\mu\gamma^2$. Further the drift speed is $O(\mu^{-1}\gamma^{-2})$; this estimate is sharp as long as a_1^0 is disjoint from 0. So again I have the cyclotron movement with angular velocities $\asymp\mu|x_1|$ and μ and the drift movement mainly along x_2.

On the other hand, if a trajectory $(x(t),\xi(t))$ on an energy level $\le c$ starts from the point $(x(0),\xi(0))$ with $x(0)\in B(0,\frac{1}{2})\cap\{x:|x_1|\le C\mu^{-1/2}\}$ then $x(t)\in B(0,\frac{1}{2})\cap\{x:|x_1|\le C^2\mu^{-1/2}\}$ as $|t|\le\epsilon$ and the averaged propagation speed is $O(1)$; this estimate is sharp for the "typical" (in the heuristic sense) trajectory. In this case there are no separate cyclotron and drift movements along $\mathbb{K}_1$ but the cyclotron movement along $\mathbb{K}_2$ remains.

Example 1.4. Assume that in some coordinates $(x_1,\dots,x_4)$ 2-form ω is defined by (1.6) and also $g^{jk}=\delta_{jk}$. Let $V=\mathsf{const}$. I am going to show that the conclusion of the previous example remains true (with the drift directed mainly along magnetic lines). Now however the angle between $\mathbb{K}_1(x)$ and $T_x\sigma$ is of magnitude r and therefore to prevent cyclotron movements from hitting Σ it is sufficient to assume that $C\mu^{-1}\gamma^{-1}r\le\gamma$ and to keep r of the same magnitude along trajectory it is sufficient to assume that $r\ge C\mu^{-1}\gamma^{-1}$. Alternatively I can assume that $r\le C\mu^{-1}\gamma^{-1}$ and then it will remain this way and the first inequality should be fulfilled with $r=C\mu^{-1}\gamma^{-1}$. So I get zone

$$\mathcal{Z}_{\mathsf{out}}=\bigl\{C\mu^{-1}\gamma^{-1}\le r\le\epsilon\mu\gamma^2\bigr\}\cup\bigl\{r\le C\mu^{-1}\gamma^{-1},\,C\mu^{-1}\gamma^{-1}\le\epsilon\mu\gamma^2\bigr\}=\\ \bigl\{\gamma\ge C\max(\mu^{-2/3},\mu^{-1/2}r^{1/2})\bigr\}. \tag{1.11}$$

1.2.2 Let instead of rather heuristic arguments above apply precise calculations. With ω defined by (1.6) and $g^{jk}=\delta_{jk}$

$$\begin{aligned} a&=\frac{1}{2}\Bigl(\xi_1^2+\xi_r^2+W(x_1,r)\Bigr),\\ W(x_1,r)&=\Bigl(\xi_2-\mu(x_1-\frac{1}{2}r^2)\Bigr)^2+r^{-2}\Bigl(\xi_\theta-\mu(x_1-\frac{1}{2}r^2)r^2\Bigr)^2=\\ &(r^2+1)\Bigl(\mu\bigl(x_1-\frac{1}{2}r^2\bigr)-\frac{\xi_\theta-\xi_2r^2}{r^2+1}\Bigr)^2+\frac{1}{r^2(r^2+1)}\bigl(\xi_\theta-\xi_2r^2\bigr)^2. \end{aligned} \tag{1.12}$$

Then ξ_θ and ξ_2 are movement integrals and the dynamics is restricted to the zone $\mathcal{Y}_{\xi_2,\xi_\theta}=\bigl\{(x_1,r):W(x_1,r)\le c\bigr\}$. While W is nonnegative and convex in the vicinity of $(0,0)$ it is a degenerate as $\xi_2=\xi_\theta=0$: $W=\mu^2(1+r^2)\bigl(x_1-\frac{1}{2}r^2\bigr)^2$ and so particle seems to move far away if time is unbounded. So I will need to bound time (but this bound will be large enough to sustain a nice remainder estimate).

Now I can consider rigorously the general case.

Let $\Bbbk_\alpha = (\Bbbk_\alpha^m)$ be eigenvectors of $(F^j_{,k}) = (g^{jl})(F_{lk})$ corresponding to eigenvalues if_α $(\alpha = 1, 2)$; then $\Bbbk^\dagger_\alpha$ are eigenvectors corresponding to eigenvalues $-if_\alpha$:

$$\sum_k F^j_{,k}\Bbbk^k_\alpha = if_\alpha \Bbbk^j_\alpha, \qquad \sum_k F^j_{,k}\Bbbk^{\dagger k}_\alpha = -if_\alpha \Bbbk^{\dagger j}_\alpha. \tag{1.13}$$

I normalize them so that

$$\sum_{j,k} g_{jk}\Bbbk^j_\alpha \Bbbk^k_\beta = 0, \qquad \sum_{j,k} g_{jk}\Bbbk^{\dagger j}_\alpha \Bbbk^{\dagger k}_\beta = 0, \qquad \sum_{j,k} g_{jk}\Bbbk^j_\alpha \Bbbk^{\dagger k}_\beta = 2\delta_{\alpha\beta}. \tag{1.14}$$

where the second equations in (1.13),(1.14) follow from the first ones.

(1.15) One can select $\Bbbk_j \in C^\infty$ satisfying (1.13), (1.14) in domain $\Omega = B(0,1) \cap \{|x_1 \le \epsilon\}$ [4].

Really, this statement is true for $\Bbbk_2$; therefore $\mathbb{K}_1 \in C^\infty$ and $(F^j_{,k})$ transforms $\mathbb{K}_1 \in C^\infty$ into itself and is skew-symmetric with respect to (g^{jk}). Then in orthonormal real base on $\mathbb{K}_1$ $F = \begin{pmatrix} 0 & f_1 \\ -f_1 & 0 \end{pmatrix}$ with $f_1 \in C^\infty$ and one can select $\Bbbk_1 \in C^\infty$.

Let

$$Z_\alpha(x,\xi) = \sum_m \Bbbk^m_\alpha(x) p_m(x,\xi), \qquad Z^\dagger_\alpha(x,\xi) = \sum_m \Bbbk^{\dagger m}_\alpha(x) p_m(x,\xi), \tag{1.16}$$
$$p_m = \xi_m - \mu V_m(x), \qquad \{p_j, p_k\} = -\mu F_{jk}$$

where the second equation follows from the first one. Then since

(1.17) $|p_j(x,\xi)| \le c_1$ on the energy levels $\{(x,\xi) : a(x,\xi) \le c_0\}$

I arrive to

$$\{Z_\alpha, Z_\beta\} \equiv \{Z^\dagger_\alpha, Z^\dagger_\beta\} \equiv 0, \quad \{Z_\alpha, Z^\dagger_\beta\} \equiv 2i\mu f_\alpha \delta_{\alpha\beta} \qquad \mod O(1). \tag{1.18}$$

[4] Provided I redefine temporarily f_1 so that $f_1/x_1 > 0$.

1.2.3 Further, note that corrected x_j:

$$x'_j = x_j - \mu^{-1}\sum_k \check{F}^{jk} p_k, \qquad \sum_k \check{F}^{jk}F_{kl} = \delta_{jl} \tag{1.19}$$

satisfy

$$\{x'_j, p_k\} = O(\mu^{-1}\gamma^{-2}), \tag{1.20}$$

$$\{x'_j, x'_k\} = \mu^{-1}\check{F}^{jk} + O(\mu^{-2}\gamma^{-3}). \tag{1.21}$$

Also note that

$$a^0(x,\xi) \stackrel{\mathsf{def}}{=} \frac{1}{2}\sum_{k,l} g^{kl}p_kp_l = \frac{1}{2}\Bigl(|Z_1|^2 + |Z_2|^2\Bigr) \tag{1.22}$$

and therefore

$$\{a^0, Z_\alpha\} \equiv \{a, Z_\alpha\} \equiv -i\mu f_\alpha Z_\alpha, \qquad \{a^0, Z^\dagger_\alpha\} \equiv \{a, Z^\dagger_\alpha\} \equiv i\mu f_\alpha Z^\dagger_\alpha \mod O(1) \tag{1.23}$$

and

$$\{a^0, b_\alpha\} \equiv \{a, b_\alpha\} \equiv 0 \mod O(1), \qquad b_\alpha \stackrel{\mathsf{def}}{=} |Z_\alpha|^2. \tag{1.24}$$

More precisely, (1.18) are fulfilled modulo linear forms[5] and therefore $\{a^0, b_\alpha\}$ is a cubic form[5] and $\{a, b_\alpha\}$ is a cubic form[5] plus a linear form[5]. Moreover, these cubic terms should contain at least one factor Z_1 or $Z_1^\dagger$ and also at least one factor Z_2 or $Z_2^\dagger$.

(1.25) Then correcting b_α ($\alpha = 1,2$) by cubic terms[5] plus linear terms[5], both multiplied by μ^{-1}, one can eliminate from $\{a, b_\alpha\}$ all cubic terms but those with $Z_1Z_2Z_2^\dagger$ and $Z_1^\dagger Z_2 Z_2^\dagger$ and all linear terms but those with Z_1, $Z_1^\dagger$. Then these remaining terms produce $O(\mu^{-1})$ error.

Further, correcting in the zone $\mathcal{Z}_{\mathsf{out}} = \{C\mu^{-1/2} \le |x_1| \le \epsilon\}$ by cubic and linear terms of the above type but with coefficients of the type "$x_1^{-1} \times \underline{\text{smooth function}}$" one can eliminate the rest of the terms but now instead of $O(\mu^{-1})$ error I get $O(\mu^{-1}x_1^{-2})$ error:

(1.26) In the classical dynamics one can correct $b_\alpha = |Z_\alpha|^2$ by $O(\mu^{-1}x_1^{-1})$ so that their propagation speeds would be $O(\mu^{-1}x_1^{-2})$. Therefore as long as $x(t) \in \Omega$ and $|x_1(t)| \asymp \gamma$, $|t| \le T$ the following is fulfilled:

$$|Z_\alpha(t)|^2 = |Z_\alpha(0)|^2 + O\bigl(\mu^{-1}\gamma^{-1} + \mu^{-1}\gamma^{-2}T\bigr).$$

[5] With respect to $(Z_1, Z_1^\dagger, Z_2, Z_2^\dagger)$ with smooth complex coefficients depending on x.

Similarly, consider $\{a, x_j\}$ which is a linear form[5)]. One can correct x_j in the same way arriving to

(1.27) In the classical dynamics one can correct x_j by $O(\mu^{-1}x_1^{-1})$ so that their propagation speeds would be $O(\mu^{-1}x_1^{-2})$. Therefore as long as $x(t)\in\Omega$ and $|x_1(t)|\asymp\gamma$, $|t|\le T$

$$x_j(t)=x_j(0)+O\bigl(\mu^{-1}\gamma^{-1}+\mu^{-1}\gamma^{-2}T\bigr).$$

Moreover, (1.24) and (1.25) with b_α replaced by x_j corrected yields that

$$\frac{dx}{dt}\in\mathbb{K}_1(x)+O(\mu^{-1}x_1^{-1})\qquad\text{for } x,\ \text{corrected modulo } O(\mu^{-1}x_1^{-1}). \tag{1.28}$$

Still it is not yet what I want which is to calculate $\{a, x_j\}$ modulo $O(\mu^{-1}x_1^{-1})$. However, let us consider first

$$b'_3\overset{\mathrm{def}}{=}-\operatorname{Re}\Bigl(i\mu^{-1}\{Z_1^\dagger, x_1\}\phi(x)^{-1}Z_1\Bigr)+x_1^2,\qquad \phi=f_1/x_1 \tag{1.29}$$

and note that $\{Z_1Z_1^\dagger, b'_3\}=O(\mu^{-1})$. Also note that $\{a, b'_3\}$ is a linear form of $Z_2, Z_2^\dagger$ modulo $O(\mu^{-1})$ and therefore it could be corrected by a linear form of $Z_2, Z_2^\dagger$ multiplied by μ^{-1} so that after correction $\{a, b'_3\}=O(1)$. Thus

(1.30) In the classical dynamics one can correct $b_3=x_1^2$ by $O(\mu^{-1})$ so that its propagation speed would be $O(\mu^{-1})$. Therefore $x_1^2(t)=x_1^2(0)+O\bigl(\mu^{-1}+\mu^{-1}T\bigr)$ and statements (1.26), (1.27) hold without precondition "as long as $x(t)\in\Omega$ and $|x_1(t)|\asymp\gamma$" which is fulfilled automatically for $x(0)\in B(0,\frac{1}{2})\cap\{\frac{1}{2}\gamma\le|x_1(0)|\le 2\gamma\}$, $T=\epsilon\mu\gamma^2$, $\gamma\ge\bar{\gamma}_1=C\mu^{-1/2}$.

(1.31) Also, trajectory originated in $B(0,\frac{1}{2})\cap\{|x_1|\le\gamma)$ remains in $B(0,1)\cap\{|x_1|\le C\gamma\}$ as $\gamma=\bar{\gamma}_1$, $T=\epsilon$.

Further, as long as $|x_1|\le c\gamma$ and $|\{Z_1, x_1\}|\le cr$, correction is $O\bigl(\mu^{-1}(r+\gamma)\bigr)$ and therefore $x_1^2(t)=x_1^2(0)+O\bigl(\mu^{-1}(r+\gamma)+\mu^{-1}T\bigr)$. Then I arrive to

(1.32) As long as $|\{Z_1, x_1\}|\le cr$ along trajectory, statements (1.30), (1.31) hold with $\bar{\gamma}_1=C\mu^{-1}+C\mu^{-1/2}r^{1/2}$. Moreover $|\{Z_1, x_1\}(t)|\le cr$ provided it was fulfilled (with constant $c/2$) at $t=0$ and $T=\epsilon\mu\gamma^2 r$, $r\ge\mu^{-1/2}+\gamma$.

1.2.4 Now let us return to calculation of $\{a, x_j\}$ modulo $O(\mu^{-1})$ where I will correct x_j. Any error, larger than this, comes from eliminating linear terms containing Z_1 or $Z_1^\dagger$, namely $\operatorname{Re}\bigl(\{Z_1^\dagger, x_j\}Z_1\bigr)$. This expression was eliminated by adding $\mu^{-1}f_1^{-1}\operatorname{Re}(i\{Z_1, x_j\}Z_1^\dagger)$ to x_j which in turn generates an error

$$\mu^{-1}\operatorname{Re} i\Bigl(\bigl\{a, f_1^{-1}\{Z_1, x_j\}\bigr\}Z_1^\dagger + f_1^{-1}\{Z_1^\dagger, x_j\}\bigl(i\{a, Z_1^\dagger\} - \mu f_1 Z_1\bigr)\Bigr). \tag{1.33}$$

In the expression (1.33) the part containing factors Z_2 and $Z_2^\dagger$ in the combinations Z_2, Z_2^2, $Z_2^\dagger$, $Z_2^{\dagger\,2}$ can be eliminated by the same way as above; the corresponding correction has an extra factor $O(\mu^{-1})$ and the corresponding error just acquires factor $O(\mu^{-1}x_1^{-1})$; therefore resulting error will be $O\bigl(\mu^{-2}x_1^{-3}|Z_1| + \mu^{-2}x_1^{-2}\bigr) = O\bigl(\mu^{-1} + \mu^{-1}\gamma^{-1}\rho\bigr)$ provided

$$|Z_1| \le c\rho, \qquad \gamma \ge c\mu^{-1/2}. \tag{1.34}$$

This leaves us with expression (which I do not call an error anymore)

$$\frac{1}{2}\Bigl(-\mu^{-1}x_1^{-2}\phi^{-1}\{Z_1Z_1^\dagger, x_1\}\operatorname{Re}\bigl(i\{Z_1, x_j\}Z_1^\dagger\bigr) + \mu^{-1}x_1^{-1}\operatorname{Re}\bigl(\alpha\{Z_1, x_j\}\bigr)|Z_2|^2\Bigr) \tag{1.35}$$

with the complex-valued coefficient $\alpha = \alpha(x')$ provided

$$f_1 = \phi(x')x_1 + O(x_1^2), \qquad x' = (x_2, x_3, x_4). \tag{1.36}$$

Note that the first term in (1.35) is equal to

$$\begin{aligned} &-\mu^{-1}x_1^{-2}\phi^{-1}\cdot\operatorname{Re}(\{Z_1, x_1\}Z_1^\dagger)\cdot\operatorname{Re}(i\{Z_1, x_j\}Z_1^\dagger) = \\ &\frac{1}{4}\mu^{-1}x_1^{-2}\phi^{-1}\cdot\operatorname{Re} i\Bigl(2\{Z_1^\dagger, x_1\}\{Z_1^\dagger, x_j\}Z_1^2 + \bigl(-\{Z_1^\dagger, x_1\}\{Z_1, x_j\} + \{Z_1^\dagger, x_j\}\{Z_1, x_1\}\bigr)|Z_1|^2\Bigr)\end{aligned} \tag{1.37}$$

and one needs to calculate coefficients at Z_1^2, $|Z_1|^2$ as $x_1 = 0$ only. Thus I need to calculate $\mathbb{k}_1$ as $x_1 = 0$. Let us decompose F_{jk}, g_{jk} and k_1 into powers of x_1; then $F_{jk}\mathbb{k}_1 = if_1\mathbb{k}_1$ implies that

$$F_{jk}^0\mathbb{k}_1^0 = 0, \qquad (F_{jk}^1 - i\phi g_{jk}^0)\mathbb{k}_1^0 + F_{jk}^0\mathbb{k}_1^1 = 0, \tag{1.38}$$

where F_{jk} corresponds to symplectic form (1.6) and therefore

$$F = \begin{pmatrix} 0 & 1 & -x_4 & x_3 \\ -1 & 0 & x_3 & x_4 \\ x_4 & -x_3 & 0 & 2\bigl(x_1 - \frac{1}{2}(x_3^2 + x_4^2)\bigr) \\ -x_3 & -x_4 & -2\bigl(x_1 - \frac{1}{2}(x_3^2 + x_4^2)\bigr) & 0 \end{pmatrix}. \tag{1.39}$$

Then (1.38), (1.39) imply for the "differentiation" part of Z_1

$$Z_{1,\,\mathrm{diff}}|_{x_1=0} = \alpha\Big(\big(r^{-1}\partial_\theta - r\partial_2\big) + \beta\big(r\partial_1 + \partial_r\big)\Big) \tag{1.40}$$
$$\text{with}\quad c^{-1}\le |\alpha|\le c,\ c^{-1}\le \operatorname{Re} i\beta \le |\beta|\le c.$$

Then

$$\operatorname{Re} i\big(\{Z_1^\dagger, x_1\}Z_1\big)\big|_{x_1=0} = 2|\alpha|^2\operatorname{Re}(i\beta)\big(\partial_\theta - r^2\partial_2\big)$$

is the differentiation along magnetic lines. Further, without any loss of the generality one can assume that $\alpha = 1$.

1.2.5 Consider first $\psi = x_1 - r^2/2$ which is a regular function of x; note that $Z_1\psi = O(x_1)$; therefore ψ can be corrected by $O(\mu^{-1})$ so that $\frac{d}{dt}\psi = O(\mu^{-1})$ and therefore original ψ is preserved modulo $O\big(\mu^{-1}(T+1)\big)$. Since I already know that x_1 is preserved modulo $O\big(\mu^{-1}\gamma^{-1}r(T+1)\big)$ I conclude that

Proposition 1.5. *Consider classical dynamics in $\Omega = B(0,1)\cap\{|x_1|\le C\epsilon\}$, originated in $\Omega' = B(0,\frac{1}{2})\cap\{|x_1|\le\epsilon\}$.*

(i) If in the original point $|x_1| = \gamma$ with $\gamma \ge C\max(\mu^{-1}r^{-1}, r^2)$ then both $|x_1(t)|$ and $r(t)$ remain of the same magnitudes as $T = \epsilon\mu\gamma r$.

(ii) If in the original point $r^2 \ge \gamma \ge C\mu^{-1/2}r^{1/2}$ then both $|x_1(t)|$ and $r(t)$ remain of the same magnitudes as $T = \epsilon\mu\gamma^2 r^{-1}$.

(iii) If in the original point $\gamma \ge C\mu^{-2/3}$ and $r \le c\mu^{-1}\gamma^{-1}$ then $|x_1(t)|$ remains of the same magnitude while $r(t) \le 2c\mu^{-1}\gamma^{-1}$ as $T = \epsilon$.

(iv) If in the original point $|x_1| \le \gamma = c\mu^{-1/2}r^{1/2}$ and $r \ge C\mu^{-1/3}$ then $r(t)$ remains of the same magnitude while $|x_1(t)| \le C\gamma$ as $T = \epsilon$.

(v) If in the original point $|x_1| \le \gamma = c\mu^{-2/3}$ and $r \le c\mu^{-1/3}$ then $r(t) \le C\mu^{-1/3}$ and $|x_1(t)| \le C\gamma$ as $T = \epsilon$.

On the other hand, at the energy level τ one can rewrite the last term in (1.35) as $\mu^{-1}x_1^{-1}\operatorname{Re}(\sigma\{Z_1, x_j\})(\tau + V)$ modulo $O(\mu^{-1} + \mu^{-1}\gamma^{-1}\rho^2)$. Consider (1.38). Using the same technique as before one can eliminate

$$\mu^{-1}x_1^{-2}\phi^{-1}\cdot\operatorname{Re} i\Big(2\{Z_1^\dagger, x_1\}\{Z_1^\dagger, x_j\}Z_1^2\Big)$$

by $\mu^{-2}x_1^{-3}\phi^{-2}\cdot\operatorname{Re}\Big(\{Z_1^\dagger, x_1\}\{Z_1^\dagger, x_j\}Z_1^2\Big)$ correction (which is $O(\mu^{-2}\gamma^{-3}r\rho^2) = O(\mu^{-1}\gamma^{-1}\rho^2)$) as $\gamma \ge \mu^{-1/2}r^{1/2}$ adding $O(\mu^{-2}\gamma^{-4}r^2\rho^2 + \mu^{-2}\gamma^{-3}\rho^2)$ error; this error is less than $\epsilon\mu^{-1}\gamma^{-2}r\rho^2$ as $\gamma \ge C\mu^{-1/2}r^{1/2}$.

Also, any Z_1 or $Z_1^\dagger$ unbalanced in the product could be treated in the same way as before.

Then I arrive to

Proposition 1.6. *As $\gamma \ge C\mu^{-1/2}r^{1/2}$ the propagation speed with respect to θ is*

$$v(\rho;x,\mu^{-1}) = \mu^{-1}\gamma^{-2}\kappa(x,\rho^2,\mu)\Bigl(\rho^2 + r^{-1}\kappa_2(x,\rho^2,\mu)\gamma\Bigr) + O\Bigl(\mu^{-2}\gamma^{-3}\rho r^{-1}\Bigr) \tag{1.41}$$

with κ,κ_2 bounded and κ disjoint from 0.

I also need to consider dynamics of $b_2 = |Z_2|^2$ more precisely. Note that

$$\bigl\{|Z_2|^2, a\bigr\} = \frac{1}{2}\operatorname{Re}\Bigl(\Bigl(\{Z_2,Z_1\}Z_1^\dagger + \{Z_2,Z_1^\dagger\}Z_1 - \{Z_2,V\}\Bigr)Z_2^\dagger\Bigr)$$

is a combination of cubic and linear terms (see footnote 5)) and the only terms where factor Z_1 is not compensated by $Z_1^\dagger$ and v.v. come from

$$\{Z_2,Z_1\} = \alpha_{11}Z_2 + \alpha_{12}Z_2^\dagger + \beta_{11}Z_1 + \beta_{12}Z_1^\dagger, \tag{1.42}$$

$$\{Z_2,Z_1^\dagger\} = \alpha_{21}Z_2 + \alpha_{22}Z_2^\dagger + \beta_{21}Z_1 + \beta_{22}Z_1^\dagger; \tag{1.43}$$

these terms are equal to $\operatorname{Re}(\sigma Z_1)|Z_2|^2$ with $\sigma = \alpha_{11}^\dagger + \alpha_{21}$. I need to calculate β as $x_1 = 0$. All other terms could be corrected by $O(\mu^{-1})$ leading to $O(\mu^{-1})$ error while this one is corrected by $\mu^{-1}x_1^{-1}\phi\operatorname{Re}(i\beta Z_1)|Z_2|^2$ leading to $O(\mu^{-1}x_1^{-2})$ error; I will calculate this latter error more precisely below. Since

$$\alpha_{11} = (2i\mu f_2)^{-1}\bigl\{\{Z_2,Z_1\},Z_2^\dagger\bigr\},$$
$$\alpha_{21} = (2i\mu f_2)^{-1}\bigl\{\{Z_2,Z_1^\dagger\},Z_2^\dagger\bigr\}$$

calculated as $Z_1 = Z_2 = 0$, I conclude that

$$\sigma = (2i\mu f_2)^{-1}\bigl(-\bigl\{\{Z_2,Z_1\},Z_2^\dagger\bigr\} + \bigl\{\{Z_2^\dagger,Z_1\},Z_2\bigr\} = -(i\mu f_2)^{-1}\bigl\{\{Z_2,Z_2^\dagger\},Z_1\bigr\}.$$

Therefore correction is $O(\mu^{-1}\gamma^{-1}\rho)$ and then the part of an error, with factors Z_2 and $Z_2^\dagger$ balanced is $O(\mu^{-1}\gamma^{-1} + \mu^{-1}\gamma^{-2}\rho r)$.

Moreover, the part of $\{|Z_2|^2, a\}$ where Z_2 and $Z_2^\dagger$ are balanced is $\kappa|Z_2|^2$ with

$$\kappa = (2\mu f_2)^{-1}\operatorname{Re}\Bigl(\bigl\{i\{Z_2,Z_2^\dagger\},Z_1\bigr\}\bigr|_{Z_1=Z_2=0}Z_1^\dagger\Bigr)$$

since $i\{Z_2, Z_2^\dagger\}$ is real; further, this expression is equal to

$$(2f_2)^{-1}\{f_2, |Z_1|^2\} + O(\rho\gamma) + O(\mu^{-1}\gamma^{-1}) = \\ (2f_2)^{-1}\{f_2, a\} - (2f_2)^{-1}\{f_2, |Z_2|^2\} + O(\rho\gamma) + O(\mu^{-1}\gamma^{-1}).$$

Note that $f_2^{-1}\{f_2, |Z_2|^2\}$ has unbalanced Z_2 or $Z_2^\dagger$ and thus is eliminated by $O(\mu^{-1})$ correction. So I arrive to an error $-(2f_2)^{-1}\{f_2, a\}|Z_2|^2$.

Thus

(1.44) (i) One can correct $|Z_2|^2$ by $O(\mu^{-1} + \mu^{-1}\gamma^{-1}\rho)$ term so that the propagation speed after correction would be $O(\mu^{-1}\gamma^{-1} + \mu^{-1}\gamma^{-2}\rho r)$.
(ii) Moreover, one can correct $f_2^{-1}|Z_2|^2$ by $(\mu^{-1} + \mu^{-1}\gamma^{-1}\rho^2)$ term so that the propagation speed after correction would be $O(\mu^{-1} + \mu^{-1}\gamma^{-2}\rho^2)$.

I will also need the shorter term classical dynamics result:

Proposition 1.7. *Let us consider dynamics described in proposition 1.5(i)-(v). Then* $\mathsf{dist}(x(t), (x(0)) \ge \epsilon\rho t$ *as* $|t| \le T_1 = \epsilon \min(\mu^{-1}\gamma^{-1}, \rho)$.

Remark 1.8. It effectively makes $T_1 = \epsilon\mu^{-1}\gamma^{-1}$ because contribution to the remainder estimate of the subzone $\{|x_1| \asymp \gamma, |Z_1| \le \rho, \rho \le \mu^{-1}\gamma^{-1}\}$ is $O(\mu h^{-3} \times \mu^{-2}\gamma^{-2} \times \gamma) = O(\mu^{-1}h^{-3}\gamma^{-1})$ which after summation over zone $\{\gamma \ge \mu^{-1/2}\}$ results in $O(\mu^{-1/2}h^{-3})$.

1.3 Quantum dynamics. I. Outer zone

In this subsection begin to deal with the quantum dynamics and the new parameter h appears; so one can now compare μ and h. Here I assume that magnetic field is rather weak; more precise requirements should vary from statement to statement but actually I do not care very much now and will just assume as needed that

$$C \le \mu \le h^{-\delta} \tag{1.45}$$

with small enough $\delta > 0$; I do not want to push it up so far.

1.3.1 As I have mentioned there are few time scales and the shortest one is $\asymp \mu^{-1}$. Let us deal with it first. Scaling $x \mapsto \mu x$, $t \mapsto \mu t$, $h \mapsto \mu h$, $\mu \mapsto 1$ I find ourselves with the standard Schrödinger operator with the propagator which is a standard Fourier integral operator. Returning to the original scale I get

Proposition 1.9. *(i) As $|t| \le T_0 = C\mu^{-1}$ propagator $e^{ih^{-1}At}$ is h-FIO corresponding to the classical dynamics $\Phi_t = e^{tH_a}$ with Hamiltonian $a(x,\xi)$. (ii) In particular, if $\psi \in C_0^\infty(B(\bar{x},\epsilon\gamma))$*

$$e^{2\pi i\mu^{-1}h^{-1}\bar{f}_2^{-1}A}\psi \equiv e^{2\pi i\mu^{-1}h^{-1}\bar{f}_2^{-1}B}\psi \tag{1.46}$$

with $\bar{f}_2 = f_2(\bar{x})$ and the standard h-pseudodifferential operator $\gamma^{-1}B$.

Proof. Let us consider classical dynamics Φ_t first; in (x,p) variables it is described by

$$\frac{dx_j}{dt} = \mu^{-1}\sum_k K_{jk}(x)p_k, \tag{1.47}$$

$$\frac{dp_j}{dt} = \sum_k L_{jk}(x)p_k + \mu^{-1}\sum_{kl} L_{jkl}(x)p_kp_l + \mu^{-1}L_j(x) \tag{1.48}$$

(after rescaling $t \mapsto \mu^{-1}t$) with uniformly smooth K_*, L_*. Therefore in these coordinates $\Phi_t : (x,Z) \mapsto \bigl(x + \mu^{-1}X(x,p,t), Y(x,p,t)\bigr)$. Thus $e^{ih^{-1}At}$ is a product of FIO corresponding to the symplectomorphism Φ_t (a standard quantization, phase function is defined in the standard way and the symbol is just 1) and some PDO.

In particular one can see easily that $\Phi_t - I = O(\gamma)$ as $t = 2\pi\bar{f}_2^{-1}$; therefore I conclude that $\Phi_t = e^{\gamma H_b}$ with some symbol $b = b(\mu^{-1}x, Z)$. Quantizing it I get that (1.46) holds but with an extra PDO factor $\mathcal{Q}$ between FIO and ψ. However then one can perturb B by operator $\gamma^{-1}hB_1$ so that

$$e^{2\pi i\mu^{-1}h^{-1}\bar{f}_2^{-1}B}\mathcal{Q} \equiv e^{2\pi i\mu^{-1}h^{-1}\bar{f}_2^{-1}(B+\gamma^{-1}hB)};$$

I leave this easy exercise to the reader. ∎

Proposition 1.10. *Let condition*

$$V \ge \epsilon_0 \tag{1.49}$$

be fulfilled. Then

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma\bigl(u\psi Q\bigr)| \le C'h^s \qquad \forall\tau : |\tau| \le \epsilon \tag{1.50}$$

as $\mu \ge K$, $\bar{T} \stackrel{\mathrm{def}}{=} Ch|\log h| \le T \le 2\pi(1-\epsilon_0)\mu^{-1}\bar{f}_2^{-1}$, $\gamma \le \epsilon$ where ψ is supported in $B(\bar{x},\gamma)$, γ is calculated in $\bar{x}$, Q is arbitrary h-pdo [6]*.*

[6] In our usual manner here and until the end of the paper $\chi \in C_0^\infty([-1,-\frac{1}{2}] \cup [\frac{1}{2},1])$, $\bar{\chi} \in C_0^\infty([-1,1])$ are functions of [BrIvr] type, $\chi_T(t) = \chi(t/T)$ etc. and $\Gamma_x(u) = u(x,x)$, $\Gamma u = \int u(x,x)\,dx$.

Now let us consider in the outer zone the intermediate scale dynamics:

Proposition 1.11. *Consider* $\bar{x} \in \mathcal{Z}_{\mathsf{out}} \overset{\mathrm{def}}{=} \{\gamma \ge C\mu^{-1/2}\}$. *Then as* $|t| \le C\mu^{-1}\gamma^{-1}$ $e^{ih^{-1}At}$ *is* h*-FIO corresponding to the classical dynamics* $\Phi_t = e^{tH_a}$ *with Hamiltonian* a *and*

$$e^{ih^{-1}tA}\psi \equiv e^{ih^{-1}t''A}e^{ih^{-1}t'B}\psi \tag{1.51}$$

where $t' = 2\pi\mu^{-1}\bar{f}_2^{-1}n$, $n = \lfloor (2\pi)^{-1}\bar{f}\mu t \rfloor$, $t'' = t - t'$.

Proof. Formula (1.51) follows from (1.46) and the fact that $|x_1|$ retains its magnitude in the classical dynamics in the zone $\mathcal{Z}_{\mathsf{out}}$. □

Proposition 1.12. *(i) Let* $\mu \le \epsilon h^{-1}|\log h|^{-1}$, $\bar{x} \in \mathcal{Z}_{\mathsf{out}}$ *and* Q *be* h*-PDO with the symbol supported in* $\{|Z_1| \ge \rho\}$ *with*

$$\rho \ge C\mu^{-1}\gamma^{-1} + C(\mu h\gamma|\log h|)^{1/2}. \tag{1.52}$$

Then (*1.50*) *holds as* $C\rho^{-2}h|\log h| \le T \le T_1 \overset{\mathrm{def}}{=} \epsilon_1\mu^{-1}\gamma^{-1}$.
(ii) In particular, let

$$\rho \ge C\mu^{-1}\gamma^{-1} + C(\mu h|\log h|)^{1/2}. \tag{1.53}$$

Then estimate (*1.50*) *holds as* $\bar{T} = Ch|\log h| \le T \le \epsilon_1\mu^{-1}\gamma^{-1}$.

Proof. Let us temporarily direct $\mathbb{K}_1$ as a coordinate plane[7] $\{x_1, x_2\}$ and consider ρ-admissible partition in (ξ_1, ξ_2). Then one can see easily that the propagation along $\mathbb{K}_1$ has the speed $\asymp \rho$. Further, condition (1.52) ensures that ρ retains its magnitude since the propagation speed of $|Z_1|^2$ does not exceed $C_0\mu^{-1}\rho^2\gamma^{-2} + C_0\mu^{-1}$ and $|Z_1|^2$ is corrected by $O(\rho\mu^{-1})$. Furthermore, the last term in (1.52) ensures the logarithmic uncertainty principle.

Then for time T the shift along $\mathbb{K}_1$ is exactly of magnitude ρT and the logarithmic uncertainty principle $\rho T \times \rho \ge Ch|\log h|$ is fulfilled as long as $T \ge T' = C\rho^{-2}h|\log h|$. This implies (i).

Now it follows from (i) and proposition 1.10 that estimate (1.50) holds as $\bar{T} = Ch|\log h| \le T \le T_1 = \epsilon\mu^{-1}\gamma^{-1}$ provided $T' \le \epsilon\mu^{-1}$ which means upgrade of condition (1.52) to (1.53). □

[7] For $T \le c\mu^{-1}\gamma^{-1}$ and $\gamma \ge C\mu^{-1/2}$ with $C = C(\epsilon, c)$ our propagation is confined to $B(\bar{x}, \epsilon\gamma)$.

1.3.2 On the other hand one can see easily that the contribution to the remainder estimate of the rather thin subzone where $|x_1|\asymp \gamma$ and condition (1.53) is violated does not exceed $C\mu h^{-3}\bigl(\mu^{-2}\gamma^{-2}+\mu h|\log h|\bigr)\gamma$; further, summation of this expression over $\mathcal{Z}_{\mathsf{out}}$ results in $C\mu^{-1/2}h^{-3}+C\mu^2h^{-2}|\log h|$. Therefore I have proven

Proposition 1.13. *Contribution to the remainder estimate of the subzone of $\mathcal{Z}_{\mathsf{out}}$ where condition (1.52) is violated is $O\Bigl(\mu^{-1/2}h^{-3}+\mu^2h^{-2}|\log h|\Bigr)$. In particular this contribution is $O(\mu^{-1/2}h^{-3})$ as $\mu\le C(h|\log h|)^{-2/5}$.*

Actually, one can get rid off logarithmic factors here but under condition (1.45) it is not needed; this estimate (without logarithmic factor) is the best possible if no non-degeneracy condition is assumed.

From now on I will consider only the *main subzone* of $\mathcal{Z}_{\mathsf{out}}$ where condition (1.53) is fulfilled and thus estimate (1.50) holds with $\bar{T}\le T\le T_1$. I need to estimate its contribution to the remainder estimate.

First of all one needs the quantum version of the results of the previous subsection.

Proposition 1.14. *Let condition (1.45) be fulfilled. Let function ψ be properly supported in $B(\bar{x},\gamma')$, and Q be properly supported in $\{|Z_1|\le\rho\}$ with $\rho\ge C_0\mu^{-1/2}$ and*

$$\gamma'\ge C\rho^{-1}h|\log h|. \tag{1.54}$$

Further, let $\gamma\ge C_0\mu^{-1/2}$ and $T\le T_2\overset{\mathrm{def}}{=}\epsilon\mu\gamma^2$ and let $x''=(x_3,x_4)$. Then
(i) If

$$r=|\bar{x}''|\ge r'\overset{\mathrm{def}}{=}C_0\mu^{-1}\gamma^{-1}+C_0\gamma' \tag{1.55}$$

then estimate

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)(1-\psi_1)(u\psi_yQ_y^t)|\le Ch^s \tag{1.56}$$

holds provided $\psi_1=1$ in the domain

$$\Bigl\{x:\,C_0^{-1}\gamma\le|x_1|\le C_0\gamma,\;\;C_0^{-1}r\le|x''|\le C_0r,\\ \mathsf{dist}(x,\bar{x})\ge C_0\gamma'+C_0\mu^{-1}\gamma^{-1}+C_0\mu^{-1}(r+\gamma)\gamma^{-2}T\Bigr\}. \tag{1.57}$$

(ii) On the other hand, if condition (1.55) is violated then estimate (1.56) holds provided $\psi_1=1$ in the domain

$$\Bigl\{x:\,C_0^{-1}\gamma\le|x_1|\le C_0\gamma,\;\;|x''|\le C_0r',\\ \mathsf{dist}(x,\bar{x})\ge C_0\gamma'+C_0\mu^{-1}\gamma^{-1}+C_0\mu^{-1}(r'+\gamma)\gamma^{-2}T\Bigr\}. \tag{1.58}$$

Proof. Proof is the standard one (see e.g. [BrIvr]) based on the analysis of the symbol

$$\varpi\Bigl(\frac{1}{T}\bigl(t\pm \epsilon v^{-1}\bigl(X(x,\xi)-X(y,\xi)\bigr)\bigr)\Bigr) \tag{1.59}$$

where ϖ is the same function as in [BrIvr], $X(x,\xi)$ is one of the corrected symbols x_1^2, $x_1-\frac{1}{2}|x''|^2$ and x_j with $j=1,\dots,4$ and v is the corresponding speed, namely μ^{-1}, μ^{-1} and $\mu^{-1}x_1^{-2}r$ respectively.

Obviously, it is sufficient to consider T such that the last term in definitions of (1.57),(1.58) is dominant: $\mu^{-1}(r+\gamma)\gamma^{-2}T\ge \gamma'+\mu^{-1}\gamma^{-1}$. One can see easily that under this condition and (1.45) symbol (1.59) is quantizable; here also one can be more specific about exponent $\delta>0$ in condition (1.45). ◻

Now I want to be more precise for smaller r, ρ:

Proposition 1.15. *In frames of proposition 1.14 as $T\le \epsilon\mu\gamma^2\rho$ estimate*

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)(1-Q_1)\psi_1(u\psi_y Q_y^t)|\le Ch^s \tag{1.60}$$

holds provided $Q_1=I$ in the domain $\{c^{-1}\rho\le |Z_1|\le c\rho\}$ and estimate (1.56) holds provided $\psi_1=1$ in the domain

$$\Bigl\{x:\,C_0^{-1}\gamma\le |x_1|\le C_0\gamma,\ C_0^{-1}r\le |x''|\le C_0 r,\\ \operatorname{dist}(x,\bar{x})\ge C_0\gamma'+C_0\mu^{-1}\gamma^{-1}+C_0\mu^{-1}\gamma^{-2}(r\rho^2+\gamma)T\Bigr\} \tag{1.61}$$

as $r\ge r'$, and in the domain

$$\Bigl\{x:\,C_0^{-1}\gamma\le |x_1|\le C_0\gamma,\ |x''|\le C_0 r',\\ \operatorname{dist}(x,\bar{x})\ge C_0\gamma'+C_0\mu^{-1}\gamma^{-1}+C_0\mu^{-1}\gamma^{-2}(r'\rho^2+\gamma)T\Bigr\}. \tag{1.62}$$

as $r\le r'$.

Proof. Proof is standard, based on the symbol (1.59) which is quantizable due to assumption (1.45); here again X is one of the corrected symbols $|Z_1|^2$, x' and v is the corresponding upper bound for a speed, namely $\mu^{-1}\gamma^{-2}\rho$, $\mu^{-1}\gamma^{-2}(\rho^2+\gamma)$ respectively. ◻

Note that under condition (1.45) $r'\asymp \mu^{-1}\gamma^{-1}$ and I want to eliminate the corresponding subzone from the future analysis. Really, let us note that the contribution of the subzone

$\{\gamma \le |x_1| \le 2\gamma, |x''| \le r, |Z_1| \le \rho\}$ to the remainder estimate is $O(T_1^{-1}h^{-3}r^2\rho^2\gamma) = O(\mu h^{-3}r^2\rho^2\gamma^2)$ (since I already eliminated subzone where condition (1.53) is violated) and plugging $r\rho = \mu^{-1}\gamma^{-1}$ results in $O(\mu^{-1}h^{-3})$. Then summation with respect to γ results in $O(\mu^{-1}h^{-3}|\log\mu|)$. So, I conclude that

(1.63) Contribution to the remainder estimate of $\mathcal{Z}_{\mathsf{out}} \cap \{r\rho \le C\mu^{-1}\gamma^{-1}\}$ is $o(\mu^{-1/2}h^{-3})$.

1.3.3 From now on I restrict myself to the analysis of subzone $\mathcal{Z}_{\mathsf{out}} \cap \{r\rho \ge C\mu^{-1}\gamma^{-1}\}$.

Now one can see that in $\mathcal{Z}_{\mathsf{out}}$ function $v(\rho; x, \mu^{-1})$ defined by (1.41) has no more than one root $\rho \ge C\mu^{-1}r^{-1}\gamma^{-1}$; I denote it by $w = w(x, \mu^{-1})$. Otherwise (if such root does not exist) I set $w = 0$. One can see easily that

$$v(\rho; x, \mu^{-1}) \asymp \mu^{-1}\gamma^{-2}(\rho^2 - w^2). \tag{1.64}$$

If $\rho - w \asymp \Delta$ then the propagation speed for corrected symbol θ is $v(\rho, .)$; then the linear shift for time $\epsilon\mu^{-1}\gamma^{-1}$ is $\asymp r\mu^{-2}\gamma^{-3}\rho\Delta$ (where r appears because θ is an angle). This shift is observable if the logarithmic uncertainty principle $r\mu^{-2}\gamma^{-3}\rho\Delta \times \Delta \ge Ch|\log h|$ holds[8)] i.e.

$$\Delta \ge C\mu\bigl(h|\log h|\bigr)^{1/2}\rho^{-1/2}\gamma^{3/2}r^{-1}. \tag{1.65}$$

Since the propagation speed of symbol $|Z_1|^2$ corrected is $O(\mu^{-1}\gamma^{-2}\rho^2)$, the magnitude of Δ is preserved during the time interval $T'' = \epsilon\mu\gamma^2\rho^{-1}\Delta$ which is larger than $C\mu^{-1}\gamma^{-1}$ as

$$\Delta \ge C\mu^{-2}\rho^{-1}\gamma^{-3}. \tag{1.66}$$

This condition is stronger than (1.65) as $r \ge \mu^{-1}$ and (1.45) is fulfilled with small enough $\delta > 0$.

To justify this assertion properly one needs to operate with the quantizable symbols and one can see easily that this is the case under assumptions (1.45) with small enough $\delta > 0$ and (1.65). Then the contribution to the remainder estimate of the corresponding subzone (where magnitudes of γ, ρ, r and Δ are fixed) does not exceed $Cr^2h^{-3}\rho\gamma\Delta/T'' = C\mu^{-1}h^{-3}r^2\rho^2\gamma^{-1}$.

Let us note that as magnitudes of γ, r, ρ are fixed and Δ ranges from $C\mu^{-2}\rho^{-1}\gamma^{-3}$ dictated by (1.66) to $\epsilon\rho$, the number of such elements does not exceed $C\log\mu$. So the summation with respect to this zone results in $C\mu^{-1}h^{-3}r^2\rho^2\gamma^{-1}\log\mu$. Further, as $\Delta \ge \epsilon\rho$,

[8)] Now Δ is the scale in ρ.

there are no more than C such elements and therefore the contribution to the remainder estimate of the part of the zone

$$\mathcal{Y}_{\gamma,\rho,r} \overset{\mathsf{def}}{=} \bigl\{\frac{1}{2}\gamma \le |x_1| \le 2\gamma,\ \frac{1}{2}\rho \le |Z_1| \le 2\rho,\ \frac{1}{2}r \le |x''| \le 2r\bigr\} \tag{1.67}$$

where condition (1.66) is fulfilled does not exceed $C\mu^{-1}h^{-3}r^2\rho^2\gamma^{-1}\log\mu$ as well. Summation with respect to ρ with $\rho \le C(\gamma/r)^{1/2}$ results in $C\mu^{-1}h^{-3}r\log\mu$ and then the summation with respect to all r,γ results in $\mu^{-1}h^{-3}|\log\mu|^2 = o(\mu^{-1/2}h^{-3})$. So, contribution to the remainder estimate of the part of the zone $\mathcal{Z}_{\mathsf{out}}$ where $r \ge \mu^{-1}\gamma^{-1}$, $\rho^2 \le C\gamma$ and condition (1.66) is fulfilled is $O(\mu^{-1/2}h^{-3})$.

The same arguments one can apply as $\rho^2 \ge C\gamma/r$ but in this case $\Delta = \epsilon\rho$ and condition (1.64) is fulfilled automatically, there will be no factor $\log\mu$ and the summation of $C\mu^{-1}h^{-3}r^2\rho^2\gamma^{-1}$ with respect to ρ, r, γ results in $C\mu^{-1/2}h^{-3}$.

Therefore I arrive to

(1.68) Contribution to the remainder estimate of the subzone of $\mathcal{Z}_{\mathsf{out}}$ where condition (1.66) is fulfilled is $O(\mu^{-1/2}h^{-3})$.

Finally, contribution to the remainder estimate of the subzone $\mathcal{Y}_{\gamma,\rho}$ with $\gamma \ge C\mu^{-1/2}$ where condition (1.66) is violated does not exceed

$$C\bigl(h^{-3}\rho\gamma \times \mu^{-2}\rho^{-1}\gamma^{-3} \times T_1^{-1}\bigr) = C\mu^{-1}\gamma^{-1}h^{-3}$$

since $T_1 = \epsilon\mu^{-1}\gamma^{-1}$ and summation with respect to γ results again in $O(\mu^{-1/2}h^{-3})$.

Combining with (1.68) I obtain immediately

Proposition 1.16. *Under condition* (*1.44*) *the contribution to the remainder estimate of the zone* $\mathcal{Z}_{\mathsf{out}}$ *is* $O\bigl(\mu^{-1/2}h^{-3}\bigr)$.

1.4 Quantum dynamics. II. Inner zone

Now I need to analyze the contribution to the remainder estimate of the *inner zone* $\mathcal{Z}_{\mathsf{inn}} = \{|x_1| \le \bar{\gamma}_0 = C\mu^{-1/2}\}$ under condition (1.45). Both classical and quantum dynamics are confined to this zone as $T \le \epsilon$ and it starts in $B(0,\frac{1}{2})$; propagation speed with respect to x does not exceed C_0. Note that the correction procedure still works as long as one eliminates only unbalanced factors Z_2, $Z_2^\dagger$.

1.4.1 Contribution to the remainder estimate of subzone $\{|Z_1|\le\rho, |x_1|\le\gamma\}$ does not exceed $CT^{-1}h^{-3}\rho^2r^2\gamma = C\mu h^{-3}\gamma\rho^2$ with $T=T_0=\epsilon\mu^{-1}$; plugging $\gamma=c\mu^{-1/2}$ and $\rho=c\mu^{-1/2}$ I get $O(\mu^{-1/2}h^{-3})$ and therefore one needs to consider only the contribution to the remainder estimate of the subzone $\mathcal{Z}_{\mathsf{inn}}\cap\{|Z_1|\ge C\mu^{-1/2}\}$; but then proposition 1.12 remains true here and one can increase $T=T_0$ to $T=T_1=\epsilon\mu^{-1/2}$ [9].

Then the contribution to the remainder estimate of zone $\{|Z_1|\le\rho, |x_1|\le\gamma, |x''|\le r\}$ does not exceed $CT^{-1}h^{-3}\rho^2r^2\gamma = C\mu^{1/2}h^{-3}\gamma r^2\rho^2$ with $T=\epsilon\mu^{-1/2}$; in particular, the total contribution of the whole inner zone $\mathcal{Z}_{\mathsf{inn}}$ is $O(h^{-3})$ and one needs to recover factor $\mu^{-1/2}$.

To recover this factor one needs just to increase $T=T_1=\epsilon\mu^{-1/2}$ to $T=T_2=\epsilon$. Before doing this just note that the contributions to the remainder estimates of subzones $\mathcal{Z}_{\mathsf{inn}}\cap\{|Z_1|\le c\mu^{-1/4}\}$, $\mathcal{Z}_{\mathsf{inn}}\cap\{|x''|\le c\mu^{-1/4}\}$ and $\mathcal{Z}_{\mathsf{inn}}\cap\{|Z_1|\cdot|x''|\le c\mu^{-1/4}|\log\mu|^{-1}\}$ are $O(\mu^{-1/2}h^{-3})$ and therefore

(1.69) One needs to consider only the contribution to the remainder estimate of the subzone $\mathcal{Z}_{\mathsf{inn}}\cap\{|x''|\ge c_1\mu^{-1/4},\ |Z_1|\ge c_1\mu^{-1/4},\ |Z_1|\cdot|x''|\ge c_1\mu^{-1/4}|\log\mu|^{-1}\}$.

I remind that the classical dynamics which starts at $\{|x_1|\le\gamma, |x''|=r\ge C_1\mu^{-1/4}\}$ remains in $\{|x_1|\le C\gamma, \bigl||x''|-r\bigr|\le C_1\mu^{-1/2}r^{-1}\}$ as $T=\epsilon$; actually it could be larger as $\rho r\ll 1$.

Repeating arguments of the proof of proposition 1.14 one can justify it easily for quantum propagation as well:

Proposition 1.17. *Under condition* (1.45) *with small enough* $\delta>0$ *the quantum dynamics which starts at* $\{|x_1|\le\gamma, |x''|=r\ge C_1\mu^{-1/4}\}$ *remains in* $\{|x_1|\le C\gamma, \bigl||x''|-r\bigr|\le C_1\mu^{-1/2}r^{-1}\}$ *as* $T=\epsilon$.

Now let us consider again function $\frac{1}{2}x_1^2$; then

$$\frac{1}{2}\{a,x_1^2\} = \mathsf{Re}\Bigl(\{Z_1^\dagger,x_1\}x_1Z_1+\{Z_2^\dagger,x_1\}x_1Z_2\Bigr) \tag{1.70}$$

and thus should be corrected by

$$\mu^{-1}\,\mathsf{Re}\,i\Bigl(\{Z_1^\dagger,x_1\}f_1^{-1}x_1Z_1+\{Z_2^\dagger,x_1\}f_2^{-1}x_1Z_2\Bigr) \tag{1.71}$$

which is $O(\mu^{-1}r)$ as $r\ge\mu^{-1/2}$ and leads to the new error

$$\begin{aligned}\mu^{-1}\,\mathsf{Re}\,i\Bigl(&\bigl\{a,\{Z_1^\dagger,x_1\}f_1^{-1}x_1\bigr\}Z_1+\{Z_1^\dagger,x_1\}f_1^{-1}x_1\bigl(\{a,Z_1\}+i\mu Z_1\bigr)+\\ &\bigl\{a,\{Z_2^\dagger,x_1\}f_2^{-1}x_1\bigr\}Z_2+\{Z_2^\dagger,x_1\}f_2^{-1}x_1\bigl(\{a,Z_2\}+i\mu Z_2\bigr)\Bigr).\end{aligned} \tag{1.72}$$

[9] As in footnote [6] one must take $T_1=\epsilon\min(\mu^{-1/2},\rho)$ but contribution to the remainder estimate of subzone $\mathcal{Z}_{\mathsf{inn}}\cap\{|Z_1|\le\mu^{-1/2}\}$ is $O(\mu h^{-3}\rho^2\gamma)=O(\mu^{-1/2}h^{-3})$.

This new error is $O(\mu^{-1})$ and thus taking $T=\epsilon r$ one can see that the oscillation of x_1^2 would be $O(\mu^{-1}r)$ which leads to

Proposition 1.18. *(i) The classical dynamics which starts at* $\{|x_1|=\gamma, |x''|=r\ge C_1\mu^{-1/4}\}$ *with* $C_0\mu^{-1/2}r^{1/2}\le\gamma\le C_0\mu^{-1/2}$ *remains in* $\{C^{-1}\gamma\le|x_1|\le C\gamma\}$ *as* $T=\epsilon r$.

(ii) The classical dynamics which starts at $\{|x_1|\le\gamma, |x''|=r\ge C_1\mu^{-1/4}\}$ *with* $\gamma=C_0\mu^{-1/2}r^{1/2}$ *remains in* $\{|x_1|\le C\gamma\}$ *as* $T=\epsilon r$.

(iii) Under condition (1.45) *with small enough* $\delta>0$ *statements (i),(ii) remain true for a quantum dynamics as well.*

So, one can see that the dynamics in the zones described in proposition 1.18(i), (ii) are different with the dynamics in the former resembling the dynamics in $\mathcal{Z}_{\mathsf{out}}$. Then the arguments of the previous subsection work perfectly in the zone described in proposition 1.18(i); tedious but easy[10)] details I leave to the reader:

Proposition 1.19. *Under condition* (1.45) *with sufficiently small* $\delta>0$ *the contribution to the remainder estimate of the zone* $\{C_0\mu^{-1/2}|x''|^{1/2}\le|x_1|\le C_0\mu^{-1/2}\}$ *does not exceed* $C\mu^{-1/2}h^{-3}$.

Thus I am left with the zone

$$\bigl\{C_1\mu^{-1/6}\le|x''|\le c,\quad |x_1|\le C_1\mu^{-1/2}|x''|^{1/2}\bigr\}. \tag{1.73}$$

Again the dynamics in the zones $\mathcal{Y}_{\gamma,\rho,r}$ with $\rho r\le\epsilon\mu\gamma^2$ and $\rho r\ge\epsilon\mu\gamma^2$ are very different with the former one more similar to the dynamics in the outer zone (as long as it remains there).

Similarly the propagation speed of the symbol $|Z_1|^2$ corrected by $O(\mu^{-1}\gamma^{-1}r|Z_1|+\mu^{-1})$ does not exceed $C\bigl(\mu^{-1}\gamma^{-2}r|Z_1|+\mu^{-1}\bigr)$ and therefore the propagation speed of $|Z_1|$ does not exceed $C\bigl(\mu^{-1}\gamma^{-2}r+\mu^{-1/2}\bigr)$ and thus magnitude of $|Z_1|\asymp\rho$ is preserved on the time interval $T=\epsilon\bigl(\mu^{-1}\gamma^{-2}r+\mu^{-1/2}\bigr)^{-1}\rho$ as long as $\gamma\ge C(\mu r\rho)^{1/2}$. As a result I arrive to

Proposition 1.20. *(i) The classical dynamics which starts at* $\{|x_1|=\gamma, |x''|=r\ge C_1\mu^{-1/4}, |Z_1|=\rho\}$ *with* $C_0\mu^{-1/2}r^{1/2}\rho^{1/2}\le\gamma\le C_0\mu^{-1/2}$ *remains in* $\{C^{-1}\rho\le|Z_1|\le C\rho\}$ *as* $T=\epsilon\min\Bigl(r,\mu\gamma r\rho^{-1},\mu^{-1}\rho^{-1}\Bigr)$.

[10)] Because one needs to worry only about eliminating factor $\mu^{1/2}$ while in the previous subsection the offending factor was $\mu^{3/2}$. One can just take $\Delta=C\mu^{-1/2}$. Also one can assume that $|x_1|\ge\hat{\gamma}=c_1\mu^{-3/5}$; otherwise $|x''|\le r=c_2\mu^{-1/5}$ and the contribution to the remainder estimate of this zone would be $O(\mu h^{-3}\gamma^2r^2)=O(\mu^{-1/2}h^{-3})$.

But then $\rho\ge\mu^{-1}\gamma^{-1}$ in the zone of interest and one can finally upgrade $T_1=\epsilon\mu^{-1/2}$ to $T_1=\epsilon\mu^{-1}\gamma^{-1}$ and the crude estimate to $O(\mu h^{-3}\rho^2\gamma^2r^2)$.

(ii) The classical dynamics which starts at $\{|x_1|\le \gamma, |x''|=r\ge C_1\mu^{-1/4}, |Z_1|=\rho\}$, *with* $\gamma=C_0\mu^{-1/2}r^{1/2}\rho^{1/2}$ *remains in* $\{C^{-1}\rho\le |Z_1|\le C\rho\}$ *as* $T=\epsilon\min\Bigl(r,\mu\gamma r\rho^{-1},\mu^{-1}\rho^{-1}\Bigr)$.

(iii) Under condition (1.45) *with small enough* $\delta>0$ *statements (i),(ii) remain true for a quantum dynamics as well.*

1.4.2 Then again arguments of the previous subsection work perfectly in the zone described in proposition 1.20(i); tedious but easy details (see footnote [9]) with the obvious modifications.) I leave to the reader:

Proposition 1.21. *Under condition* (1.44) *with sufficiently small* $\delta>0$ *the contribution to the remainder estimate of the zone* $\{C_0\mu^{-1/2}|x''|^{1/2}|Z_1|^{1/2}\le |x_1|\le C_0\mu^{-1/2}\}$ *does not exceed* $C\mu^{-1/2}h^{-3}$.

Therefore I am left with the *true inner zone*

$$\mathcal{Z}^0_{\mathsf{inn}}\overset{\mathsf{def}}{=}\bigl\{|x_1|\le C_1\mu^{-1/2}|x''|^{1/2}|Z_1|^{1/2}\bigr\}. \tag{1.74}$$

Similarly to (1.69)

(1.75) One needs to consider only the contribution to the remainder estimate of the subzone $\mathcal{Z}^0_{\mathsf{inn}}\cap\{|x''|\ge c_1\mu^{-1/6},\ |Z_1|\ge c_1\mu^{-1/6},\ |Z_1|\cdot|x''|\ge c_1\mu^{-1/6}|\log\mu|^{-1}\}$.

Let us consider classical dynamics in this zone first; I am interested in the time interval $T\le\epsilon r$. There

$$\{a,Z_1\}=\{Z_1^\dagger,Z_1\}Z_1+\{Z_2^\dagger,Z_1\}Z_2+\{Z_2,Z_1\}Z_2^\dagger-\{V,Z_1\}$$

and then I correct Z_1 to

$$\zeta_1=Z_1+i\mu^{-1}f_2^{-1}\{Z_2^\dagger,Z_1\}Z_2-i\mu^{-1}f_2^{-1}\{Z_2,Z_1\}Z_2^\dagger \tag{1.76}$$

satisfying

$$\begin{aligned}\{a,\zeta_1\}\equiv\{Z_1^\dagger,Z_1\}Z_1-if_2^{-1}\mu^{-1}\Bigl(\{Z_2,\{Z_2^\dagger,Z_1\}\}-\{Z_2^\dagger,\{Z_2,Z_1\}\}\Bigr)|Z_2|^2-\{V,Z_1\}=\\ \{Z_1^\dagger,Z_1\}Z_1+if_2^{-1}\mu^{-1}\{Z_1,\{Z_2,Z_2^\dagger\}\}|Z_2|^2-\{V,Z_1\}\equiv\\ \{Z_1^\dagger,Z_1\}Z_1-f_2^{-1}\{Z_1,f_2\}|Z_2|^2-\{V,Z_1\}\qquad \mod O(\mu^{-1/2}).\end{aligned}$$

Since I do not mind to change the speed to one of the same magnitude and I consider only level 0, I can assume with no loss of the generality that

$$f_2 = 1; \tag{1.77}$$

then finally $\{a, \zeta_1\} \equiv \{\zeta_1^\dagger \zeta_1 - V, \zeta_1\} \mod O(\mu^{-1/2})$.

Also corrected functions x_j satisfy $\{a, x'_j\} \equiv \{\zeta_1^\dagger\zeta_1 - V, x'_j\} \mod O(\mu^{-1/2})$ and I arrive to the following conclusion:

(1.78) Under condition (1.77) the classical dynamics in zone $\{|x_1| \le C\mu^{-1/2}\}$ in variables x_j, Z_1, Z_2 is described modulo $O(\mu^{-1} + \mu^{-1/2}T)$ by the solution to the short system

$$\frac{dZ_1}{dt} = \frac{1}{2}\{Z_1^\dagger Z_1 - V, Z_1\}, \tag{1.79}$$

$$\frac{dx_j}{dt} = \frac{1}{2}\{Z_1^\dagger Z_1 - V, x_j\}, \tag{1.80}$$

$$\{Z_1, Z_2\} = O(\mu^{-1}), \quad \{Z_1, Z_2^\dagger\} = O(\mu^{-1}). \tag{1.81}$$

Further, for each given initial point one can replace all functions modulo $(x_1 - \frac{1}{2}r^2)$, $(x_2 - \theta r^2)$ so that I get two-dimensional system in variables (r, θ) described by the Hamiltonian $\frac{1}{2}(Z_1^\dagger Z_1 - V)$ with $Z_1, Z_1^\dagger$ satisfying commutator relation

$$\{Z_1, Z_1^\dagger\} = \mu\phi\bar{r}(r - \bar{r}) + O(\mu^{-1})) \tag{1.82}$$

which is exactly system studied extensively in [Ivr6] with magnetic field $\phi(r - \bar{r})$ and coupling constant $\bar{\mu} = \mu\bar{r}$. Condition (1.74) means exactly that I am in the inner zone $\{|r - \bar{r}| \le C\bar{\mu}^{-1/2}|Z_1|^{1/2}\}$ for this system; since (in contrast to [Ivr6]) $|Z_1|$ is not disjoint from 0 anymore one needs to define inner zone in this way.

Then the drift speed with respect to θr is

$$\asymp \Bigl(\eta - k^* \phi^{-1/2}\rho\Bigr)\rho r \tag{1.83}$$

where constant $k^* \approx 0.66$ is defined in [Ivr6] and η is the corrected symbol $\mu x_1^2/2$ which leads to its propagation speed given by (1.72) multiplied by μ.

One can see easily that the fourth term there $\{Z_2^\dagger, x_1\} f_2^{-1} x_1 \bigl(\{a, Z_2\} + i\mu Z_2\bigr)$ is $O(\mu^{-1/2})$. Further, the third term $\bigl\{a, \{Z_2^\dagger, x_1\} f_2^{-1} x_1\bigr\} Z_2$ could be rewritten modulo $O(\mu^{-1/2})$ and terms with unbalanced $Z_2, Z_2^\dagger$ as $|\{Z_2, x_1\}|^2 f_2^{-1} |Z_2|^2$ and disappears after applying $\operatorname{Re} i$.

So I am left with just two terms

$$\operatorname{Re} i\Bigl(\bigl\{a, \{Z_1^\dagger, x_1\}\phi^{-1}\bigr\} Z_1 + \{Z_1^\dagger, x_1\}\phi^{-1}\bigl(\{a, Z_1\} + i\mu Z_1\bigr)\Bigr) \tag{1.84}$$

where $\phi = f_1/x_1$ and one can replace a by $Z_1^\dagger Z_1 - V$ (other terms have unbalanced $Z_2, Z_2^\dagger$).

1.4.3 So I am *exactly* in [Ivr6] situation. I remind that in [Ivr6] symbol Z_1 could be reduced to

$$Z_1 = e^{i\beta(x)}\Bigl(\xi_1 + i\alpha(x)\bigl(\xi_2 - V_2(x)\bigr)\Bigr) \tag{1.85}$$

with real-valued α, β, V_2 such that $\alpha = 1 + O(x_1),\ V_2 = x_1^2/2 + O(x_1^3)$.

One can see easily that in this case that corrected μx_1^2 would be $\xi_2 + O(\mu^{-1/2})$; I want to remind that in [Ivr6] exactly deviation of ξ_2 from $k^*\phi^{-1/2}\rho$ was used as the measure of the drift with $\rho = V^{1/2}$. Note that the potential V now should be replaced by $(V - E_2)$ with $E_2 = |Z_2^2|$.

Therefore due to the logarithmic uncertainty principle the violation of periodicity is observable after the first turn as

$$\Bigl(\eta - k^*\phi^{-1/2}\rho\Bigr)\rho r\mu^{-1/2} \times \Bigl(\eta - k^*\phi^{-1/2}\rho\Bigr) \ge Ch|\log h|; \tag{1.86}$$

under assumption (1.45) the latter condition is the automatic corollary of

$$\Delta = |\eta - k^*\phi^{-1/2}\rho| \ge C\mu^{-1/2}r^{-1/2}\rho^{-1/2}. \tag{1.87}$$

Under this condition the drift speed for the "short" system is greater than $C\epsilon_1\mu^{-1/2}$ which is the larger than the error in the drift speed $c\mu^{-1/2}$ (see (1.78).

In this case $T_1 = \epsilon r\Delta$ and the rest is easy but extra logarithmic factor would appear; however considering the right direction one can take $T_1 = \epsilon r\rho^{1/2}\Delta^{1/2}$ and following [Ivr6] with the standard justification on the quantum level one can prove easily that

(1.88) The contribution of the zone where condition (1.87) is fulfilled to the remainder estimate is $O(\mu^{-1/2}h^{-3})$.

On the other hand, contribution to the remainder estimate of zone where this condition is violated would not exceed

$$C\Bigl(\int \mu^{-1/2}h^{-3}r^{-1/2}\rho^{-1/2}rdrd\rho\Bigr) = O(\mu^{-1/2}h^{-3})$$

as well.

So, the main result of this section is proven:

Proposition 1.22. *Under condition* (*1.45*) *with small enough* $\delta > 0$ *the remainder estimate is* $O(\mu^{-1/2}h^{-3})$ *while the principal part is given by the standard Weyl formula (which is equal modulo the remainder estimate to magnetic Weyl formula).*

This statement implies trivially

Corollary 1.23. *Under conditions* (*0.2*) *and* (*1.45*) *estimate* (*0.3*) *holds.*

2 Strong magnetic field. Canonical forms

Now I want to consider the main case

$$h^{-\delta} \le \mu \le Ch^{-1} \tag{2.1}$$

with an arbitrarily small exponent $\delta > 0$. In this section I consider different canonical forms of the Magnetic Schrödinger Operator in question (depending on the zone).

2.1 Precanonical Form

2.1.1 Let

$$U_2(x,\xi) = f_2(x)^{-1/2} Z_2(x,\xi) \tag{2.2}$$

Then

$$-i\mu^{-1}\{U_2, U_2^\dagger\} \equiv 2 \qquad \mathrm{mod}\ \mathcal{O}_k \tag*{$(2.3)_k$}$$

with $k=1$ where $\mathcal{O}_k = \mathcal{O}_k(\mu^{-1}Z_1, \mu^{-1}Z_1^\dagger, \mu^{-1}Z_2, \mu^{-1}Z_2^\dagger)$ is the space of the sum of polynomials containing monoms of order k or higher with respect to $(\mu^{-1}Z_1, \mu^{-1}Z_1^\dagger, \mu^{-1}Z_2, \mu^{-1}Z_2^\dagger)$ with the coefficients, smoothly depending on x.

Note that correcting U_2 by $U_2\mathcal{O}_k$ one trades $(2.3)_k$ by $(2.3)_{k+1}$ and therefore after an appropriate correction (2.3) holds with arbitrarily large $k=M$.

So, correcting U_2:

$$U_2 \mapsto U_2 + \sum_{\substack{k+l+p+q\ge 2\\ k\ge 1}} \alpha_{klpq}(x) U_2^k U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{1-k-l-p-q} \tag{2.4}$$

one one can achieve

$$-i\mu^{-1}\{U_2, U_2^\dagger\} \equiv 2 \qquad \mathrm{mod}\ O(\mu^{-M}) \tag{2.5}$$

with arbitrarily large M.

Note that $\{U_2, Z_1\} \equiv \{U_2^\dagger, Z_1\} \equiv 0$ modulo $\mathcal{O}_1$ both originally and after correction (2.4).

I claim that

Proposition 2.1. *Correcting*

$$Z_1 \mapsto Z_1 + \sum_{\substack{k+l+p+q\ge 2\\ k+l\ge 1}} \alpha'_{klpq}(x) U_2^k U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{1-k-l-p-q}, \tag{2.6}$$

$$x_j \mapsto x_j + \sum_{k+l\ge 1} \alpha''_{jklpq}(x) U_2^k U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{-k-l-p-q}, \tag{2.7}$$

one can arrange

$$\{U_2, x_j\} \equiv \{U_2, Z_1\} \equiv \{U_2, Z_1^\dagger\} \equiv 0, \tag{2.8}$$

$$\{U_2^\dagger, x_j\} \equiv \{U_2^\dagger, Z_1\} \equiv \{U_2^\dagger, Z_1^\dagger\} \equiv 0 \qquad \mod O(\mu^{-M}). \tag{2.9}$$

Proof. Really, correcting in the way described (but with the sum over $k \geq 1$ rather than $k + l \geq 1$) I can achieve (2.9). However, *so far corrected symbol x_j is not necessarily real-valued and corrected symbols Z_1 and $Z_1^\dagger$ are not necessarily complex conjugate.*

Then due to (2.9), (2.5) and Poisson identity $\{U_2^\dagger, \{U_2, y\}\} \equiv 0$ with $y = x_j, Z_1, Z_1^\dagger$. Therefore

$$\{U_2, Z_1\} \equiv \sum_{l+p+q\geq 1} \beta_{lpq}(x) U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{1-l-p-q},$$

$$\{U_2, Z_1^\dagger\} \equiv \sum_{l+p+q\geq 1} \beta'_{lpq}(x) U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{1-l-p-q},$$

$$\{U_2, x_j\} \equiv \sum_{l+p+q\geq 1} \beta''_{jlpq}(x) U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{-l-p-q}$$

where x, Z_1 and $Z_1^\dagger$ are already corrected symbols. Since at this moment x_j are not necessarily real-valued one needs to plug them formally into functions and use a Taylor decomposition with respect to $\mu^{-1}Z_1, \mu^{-1}Z_2, \mu^{-1}U_1, \mu^{-1}U_2$.

Then one can correct x_j, Z_1 and $Z_1^\dagger$ according to formulae (2.7),(2.8) with summation $k = 0$ so that (2.8) holds.

Note that instead of Z_1 and $Z_1^\dagger$ one could consider $\operatorname{Re} Z_1$ and $\operatorname{Im} Z_1$ and correct them deriving (2.8)-(2.9). At this moment corrected symbols x_j, $\operatorname{Re} Z_1$ and $\operatorname{Im} Z_1$ are not necessarily real-valued. However U_2 and $U_2^\dagger$ are truly complex conjugate and then (2.8)-(2.9) for corrected symbols x_j, $\operatorname{Re} Z_1$, $\operatorname{Im} Z_1$ imply the same equalities for the real parts of them. Let us replace then (corrected) symbols x_j, $\operatorname{Re} Z_1$ and $\operatorname{Im} Z_1$ by their real parts. After this (corrected) symbols x_j become real-valued and (corrected) symbols $Z_1 = \operatorname{Re} Z_1 + i \operatorname{Im} Z_1$ and $Z_1^\dagger = \operatorname{Re} Z_1 - i \operatorname{Im} Z_1$ become complex conjugate. □

2.1.2 One can rewrite operator in question in these new variables as

$$2A \equiv f_2(x) U_2^\dagger U_2 + Z_1^\dagger Z_1 - V(x) + \tag{2.10}$$

$$\sum_{k+l+p+q\geq 3} b_{klpq}(x) U_2^k U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{2-k-l-p-q} + \sum_{k+l+p+q\geq 3} b'_{klpq}(x) U_2^k U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{-k-l-p-q}$$

modulo lower order terms where the first line is the "main part" and the second line is the "perturbation"; here the first part of the "perturbation" comes from $f_2 U_2^\dagger U_2 + Z_1^\dagger Z_1$ and the second part comes from $-V$.

Proposition 2.2. *For arbitrarily large M there exist $\epsilon = \epsilon(M) > 0$ and a real valued symbol*

$$\mathcal{L} = \sum_{k+l\geq 1} \ell_{kpqm}(x) U_2^k U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{-1-k-l-p-q-m} \tag{2.11}$$

such that in the strip $\{|x_1| < \epsilon\}$ the following equality holds (so far only for the principal symbols of operators in question):

$$A^\# \overset{\mathrm{def}}{=} e^{-i\mu^{-1}h^{-1}\mathcal{L}} A e^{i\mu^{-1}h^{-1}\mathcal{L}} \equiv \frac{1}{2}\Bigl(f_2(x^\#) U_2^\# U_2^{\#\dagger} + Z_1^\# Z_1^{\#\dagger} - V(x^\#)\Bigr) + \sum_{2k+p+q+2m\geq 3} B_{kpqm}(x^\#) \bigl(U_2^\# U_2^{\#\dagger}\bigr)^k Z_1^{\#\, p} Z_1^{\#\dagger\, q} \mu^{2-2k-p-q-2m} \tag{2.12}$$

where $e^{-i\mu^{-1}h^{-1}\mathcal{L}}$, $e^{i\mu^{-1}h^{-1}\mathcal{L}}$ are (formal) $\mu^{-1}h$-FIOs and

$$x_j^\# = e^{-i\mu^{-1}h^{-1}\mathcal{L}} x_j e^{i\mu^{-1}h^{-1}\mathcal{L}}, \qquad U_2^\# = e^{-i\mu^{-1}h^{-1}\mathcal{L}} U_2 e^{i\mu^{-1}h^{-1}\mathcal{L}}, \qquad Z_1^\# = e^{-i\mu^{-1}h^{-1}\mathcal{L}} Z_1 e^{i\mu^{-1}h^{-1}\mathcal{L}} \tag{2.13}$$

are linked to the original symbols x_j, U_2, Z_1 by formulae similar to (2.4), (2.6), (2.7) but containing also factors μ^{-1} exactly on the same role as U_2 or $U_2^\dagger$ so that $k+l$ in the condition of summation is replaced by $k+l+m$ [11]*.*

All equalities here hold modulo $O(\mu^{-M})$.

Proof. Proof is standard (see [Ivr5]). The problem of resonance is avoided because I remove only terms $b_*(x) U_2^k U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q}$ with $k \neq l$ which leads to the denominator

$$\mu\bigl((k-l)f_2 + (p-q)f_1\bigr)$$

---

[11] More precisely

$$U_2 \mapsto U_2^\# = U_2 + \sum_{k+l+p+q+m\geq 2} \alpha_{klpqm}(x) U_2^k U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{1-k-l-p-q-m}, \tag{2.4*}$$

$$Z_1 \mapsto Z_1^\# = Z_1 + \sum_{k+l+p+q+m\geq 2} \alpha'_{klpqm}(x) U_2^k U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{1-k-l-p-q-m}, \tag{2.6*}$$

$$x_j \mapsto x_j^\# = x_j + \sum_{k+l+p+q+m\geq 1} \alpha''_{jklpqm}(x) U_2^k U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{-k-l-p-q-m}. \tag{2.7*}$$

in the correction term; this denominator is of magnitude $\mu(k-l)$ since $k\neq l$, $f_2\asymp 1$ and $f_1\asymp |x_1|\le \epsilon(M)$. $\square$

Now one can upgrade all the above arguments to operators so that equalities (2.5),(2.8) and (2.9) hold for commutators rather than for Poisson brackets and (2.12) is replaced by

$$(2.12)^* \quad A^{\#} \overset{\mathrm{def}}{=} e^{-i\mu^{-1}h^{-1}\mathcal{L}} A e^{i\mu^{-1}h^{-1}\mathcal{L}} \equiv \frac{1}{2}\Bigl(f_2^{\mathrm{w}} U_2^* \bullet U_2 + Z_1^* \bullet Z_1 - V^{\mathrm{w}}\Bigr) + \sum_{2k+p+q+2m+2s\ge 3} B^{\mathrm{w}}_{kpqms} (U_2\bullet U_2^*)^l Z_1^p Z_1^{*\,q} \mu^{2-2k-2m-p-q-s} h^s$$

where a^{w} is the Weyl quantization of symbol a and one should remember that the corrected symbol x is not just a function of coordinates but symbols and therefore the functions of x are replaced by the symbols of $\mu^{-1}h$-PDOs. Here and below $\bullet$ means the symmetrized product: $K\bullet L = \frac{1}{2}(KL+LK)$. So, I arrive to

Proposition 2.3. *For arbitrarily large M and arbitrarily small $\delta>0$ there exist $\epsilon=\epsilon(M,\delta)>0$ and a real valued symbol*

$$(2.11)^* \qquad \mathcal{L} = \sum_{k+l+p+q+m+2s\ge 1} \ell_{kpqms}(x) U_2^k U_2^{\dagger\,l} Z_1^p Z_1^{\dagger\,q} \mu^{-1-k-l-p-q-m-s} h^s$$

such that under condition (2.1) in the strip $\{|x_1|<\epsilon\}$ formula $(2.12)^*$ holds where $e^{-i\mu^{-1}h^{-1}\mathcal{L}}$, $e^{i\mu^{-1}h^{-1}\mathcal{L}}$ are $\mu^{-1}h$-FIOs and $x_j^{\#}$, $U_2^{\#}$, $Z_1^{\#}$ are full symbols still defined by (2.13) and they are linked to the original symbols x_j, Z_1, U_2 by formulae similar to $(2.4)^*$,$(2.6)^*$,$(2.7)^*$ but containing also factors $(\mu^{-1}h)$ of the double value of U_2 or $U_2^\dagger$ or μ^{-1} so that $k+l+m$ in the condition of summation is replaced by $k+l+m+2s$ [12]*.*

All equalities here and below hold modulo $O(h^M)$.

[12] More precisely

$$(2.4)^{**} \qquad U_2 \mapsto U_2^{\#} = U_2 + \sum_{k+l+p+q+m+2s\ge 2} \alpha_{klpqms}(x) U_2^k U_2^{\dagger\,l} Z_1^p Z_1^{\dagger\,q} \mu^{1-k-l-p-q-m-s} h^s,$$

$$(2.6)^{**} \qquad Z_1 \mapsto Z_1^{\#} = Z_1 + \sum_{k+l+p+q+m+2s\ge 2} \alpha'_{klpqmm}(x) U_2^k U_2^{\dagger\,l} Z_1^p Z_1^{\dagger\,q} \mu^{1-k-l-p-q-m-s} h^s,$$

$$(2.7)^{**} \qquad x_j \mapsto x_j^{\#} = x_j + \sum_{k+l+m+2s\ge 1} \alpha''_{jklpqms}(x) U_2^k U_2^{\dagger\,l} Z_1^p Z_1^{\dagger\,q} \mu^{-k-l-p-q-m-s} h^s,$$

2.1.3 Note that due to (2.5) for $U_2^\#$ there exists $\mu^{-1}h$-FIO $\mathcal{F}$ such that

$$\mathcal{F}^*\mathcal{F}\equiv\mathcal{F}\mathcal{F}^*\equiv I \qquad \text{and}\qquad \mathcal{F}^*U_2^\#\mathcal{F}\equiv (hD_2-i\mu x_2). \tag{2.14}$$

Then

$$\mathcal{F}^*(U_2^\#\bullet U_2^{\#\dagger})\mathcal{F}\equiv\bigl(h^2D_2^2+\mu^2x_2^2\bigr) \tag{2.15}$$

and also due to (2.8),(2.9) $\mu^{-1}\mathcal{F}^*Z_1^\#\mathcal{F}$, $\mathcal{F}^*x_j^\#\mathcal{F}$ and $\mathcal{F}^*B_{kpqms}\mathcal{F}$ are $\mu^{-1}h$-PDOs with symbols depending only on x',ξ' ($x'=(x_1,x_3,x_4)$). Let us redefine $Z_1^\#\overset{\mathsf{redef}}{=}\mu^{-1}\mathcal{F}^*Z_1^\#\mathcal{F}$, $B^{\mathsf{w}}_{kpqms}\overset{\mathsf{redef}}{=}\mathcal{F}^*B^{\mathsf{w}}_{kpqms}\mathcal{F}$ and $A^\#\overset{\mathsf{redef}}{=}\mathcal{F}^*A^\#\mathcal{F}$; then also automatically $Z_1^{\#\dagger}\overset{\mathsf{redef}}{=}\mu^{-1}\mathcal{F}^*Z_1^{\#\dagger}\mathcal{F}$ is an adjoint operator. I remind that f_2^{w} and B^{w}_{kpqms} are operators with the real-valued symbols and $A^\#$ is again defined by $(2.12)^*$.

Now one can decompose u into series

$$u(x,y,t)=\sum_{n,n'\in\mathbb{Z}^+}\mu h^{-1}u_{nn'}(x',y',t)\,\Upsilon_n(x_2)\Upsilon_{n'}(y_2),\qquad \Upsilon_n(x_2)\overset{\mathsf{def}}{=}\upsilon_n\bigl(\mu^{1/2}h^{-1/2}x_2\bigr) \tag{2.16}$$

where υ_n are (real-valued and orthonormal) Hermite functions and then replace operator $\bigl(h^2D_2^2+\mu^2x_2^2\bigr)$ by $(2n+1)\mu h$, thus reducing operator $A^\#$ to the family of 3-dimensional $\mu^{-1}h$-PDOs

$$\begin{multlined}\mathcal{A}_n\overset{\mathsf{def}}{=}\frac{1}{2}\Bigl(f_2^\#(2n+1)\mu h+(Z_1^\#\bullet Z_1^{\#\dagger})-V^\#\Bigr)+\\ \sum_{2k+p+q+2m+2s\ge1}B^w_{kpqms}\times\bigl((2n+1)\mu h\bigr)^kZ_1^{\#\,p}Z_1^{\#\dagger\,q}\mu^{2-2k-2m-p-q-s}h^s;\end{multlined} \tag{2.17}$$

again the first line is the *main part* of operator and $f^\#$, $V^\#$ are transformed f_2 and V.

Remark 2.4. (i) In this and below formulae one actually needs to consider

$$n\le C_0/(\mu h)+C|\log h|. \tag{2.18}$$

Really, I am interested in the domain in the phase space where $\{a\le c\}$ or, equivalently, $\{|U_2|\le c,|Z_1|\le c\}$. Note that condition $\{|U_2|\le\varrho\}$ does not contradict to the logarithmic uncertainty principle as long as $\varrho^2\ge C\mu h|\log h|$ and thus one can take an upper bound for a as $\varrho^2=c+C\mu h|\log h|$ which implies (2.18). This estimate perfectly suits my purposes as long as

$$h^{-\delta}\le\mu\le\epsilon h^{-1}|\log h|^{-1}. \tag{2.19}$$

(ii) From the operator point of view which one needs to apply only as

$$\epsilon h^{-1}|\log h|^{-1}\le \mu \le \epsilon h^{-1} \tag{2.20}$$

in the final analysis I need to consider only $n\le C_0/(\mu h)$.

Remark 2.5. (i) Obviously $f_j^\#$, $V^\#$ and $\mu^{-1}Z_1^\#$ are $\mu^{-1}h$-PDOs with symbols $f_j\circ\Psi_2$, $V\circ\Psi_2$ and $\mu^{-1}Z_1\circ\Psi_2$ respectively where $\Psi_2:\mathbb{R}^6\ni(x',\xi')\to x\in\mathbb{R}^4$ is some smooth map. Actually I am interested in the symbols of $f_2^\#$ and $V^\#$ only modulo symbols of $\mu^{-1}Z_1^\#$, $\mu^{-1}Z_1^{\#\dagger}$ and therefore actually I am interested only in the map $\Sigma\to\mathbb{R}^4$. However at this moment I do not have a natural parametrization of Σ.

(ii) Note that $f_1^\#$ is not necessarily x_1 anymore. Still since $\nabla\operatorname{Re}Z_1$, $\nabla\operatorname{Im}Z_1$, $\nabla\operatorname{Re}Z_2$, $\nabla\operatorname{Im}Z_2$ and ∇f_1 were linearly independent[13)] this linear independence statement is true for $\nabla\operatorname{Re}Z_1^\#$, $\nabla\operatorname{Im}Z_1^\#$ and $\nabla f_1^\#$[14)] as well. Therefore after $\mu^{-1}h$-FIO transformation (in $(x',\mu^{-1}hD')$) one can achieve $f_1^\#=x_1$.

So, I arrive to

Proposition 2.6. *Let condition (2.1) be fulfilled. Then by means of $\mu^{-1}h$-transform, decomposition (2.16) and one more $\mu^{-1}h$-transform one can reduce operator to the family of 3-dimensional operators (2.17) with n satisfying (2.18) and with*

$$i\mu^{-1}\{Z_1^\#,Z_1^{\#\dagger}\}=f_1^\#=x_1\qquad \operatorname{mod}\,(Z_1,Z_1^\dagger). \tag{2.21}$$

2.2 Canonical form away from Λ

2.2.1 Let us consider first the canonical form in the subdomain, disjoint from Λ which means that

$$|\{Z_1,f_1\}|\ge\epsilon_0. \tag{2.22}$$

Let us reduce (2.17) to a canonical form (multiplied by an elliptic operator). Note that the multiplication of $Z_1^\#$ by symbol β implies the multiplication of $Z_1^{\#\dagger}$ by $\beta^\dagger$ and therefore the multiplication of $f_1^\#$ by $|\beta|^2$ modulo $\mathcal{O}_1$ and finally the multiplication of $\{Z_1^\#,f_1^\#\}$ by $|\beta|^2\beta$ modulo $\tilde{\mathcal{O}}_1$ as well where here and below $\tilde{\mathcal{O}}_m$ means the sum of monoms of the type $\alpha_{ljk}\mu^{l-m}x_1^lZ_1^{\#j}Z_1^{\#\dagger k}$ with $m\ge j+k+l$ and smooth coefficients α_{ljk}. Also to maintain (2.12) and (2.17) one needs to multiply operators A and $\mathcal{A}_n$ (and thus $f_2^\#$, $V^\#$ and all the perturbation terms) by $|\beta|^2$.

13) All the linear independence statements are uniform.

14) Where in the former case it was $\nabla_{x,\xi}$ and in the latter one $\nabla_{x',\xi'}$; I remind that $x'=(x_1.x_3,x_4)$.

Then picking up
$$\beta = \{Z_1^{\#}, f_1^{\#}\}^{\dagger}\cdot |\{Z_1^{\#}, f_1^{\#}\}|^{-4/3}$$
one can achieve
$$\{Z_1^{\#}, f_1^{\#}\} = 1+\alpha \tag{2.23}$$
with $\alpha\in\tilde{\mathcal{O}}_m$ and $m=1$. Consequently, multiplying $Z_1^{\#}$ by $(1+\beta)$ with $\alpha\in\tilde{\mathcal{O}}_m$ one changes $\{Z_1^{\#}, f_1^{\#}\}$ by $(2\beta+\beta^\dagger)\{Z_1^{\#}, f_1\}$ modulo $\tilde{\mathcal{O}}_{m+1}$ and picking $\beta=\frac{1}{3}\alpha^\dagger-\frac{2}{3}\alpha$ one can achieve (2.23) with $\alpha\in\tilde{\mathcal{O}}_M$ with arbitrarily large M (and after asymptotic summation with $M=\infty$). Unfortunately elements of $\tilde{\mathcal{O}}_M$ (in contrast to elements of $\mathcal{O}_M$) are not necessarily negligible unless $|x_1|\le h^\delta$.

However, it is easy to see that multiplying $Z_1^{\#}$ by $(1+\beta)$ with $\beta\in\mathcal{O}_m\cap\tilde{\mathcal{O}}_\infty$ one changes $\{Z_1^{\#}, f_1^{\#}\}$ by $\{Z_1^{\#},\phi\}$ with
$$\phi = f_1(\beta+\beta^\dagger)+i\mu^{-1}\{Z_1^{\#},\beta^\dagger\}-i\mu^{-1}\{Z_1^{\#\dagger},\beta\}$$
modulo $\mathcal{O}_m\cap\tilde{\mathcal{O}}_\infty$. Obviously there exists $\beta\in\mathcal{O}_1\cap\tilde{\mathcal{O}}_\infty$ eliminating error α in (2.23) modulo $\mathcal{O}_1\cap\tilde{\mathcal{O}}_\infty$. One can see easily that the only restriction to $\phi\in\mathcal{O}_{m-1}\cap\tilde{\mathcal{O}}_\infty$ is that it must be real valued.

Therefore step by step one can make $\operatorname{Re}\alpha\in\mathcal{O}_M\cap\tilde{\mathcal{O}}_\infty$: $\{\operatorname{Re}Z_1, f_1\}\equiv 1 \mod \mathcal{O}_M\cap\tilde{\mathcal{O}}_\infty$ with arbitrarily large M. But then setting $f_1^{\#}=i\mu^{-1}\{Z_1^{\#}, Z_1^{\#\dagger}\}=2\mu^{-1}\{\operatorname{Re}Z_1^{\#},\operatorname{Im}Z_1^{\#}\}$ one gets
$$\bigl\{\operatorname{Re}Z_1^{\#},\{\operatorname{Im}Z_1^{\#}, f_1\}\bigr\}=\bigl\{\operatorname{Im}Z_1^{\#},\{\operatorname{Re}Z_1^{\#}, f_1\}\bigr\}\in\mathcal{O}_{M-1}\cap\tilde{\mathcal{O}}_\infty$$
and since $\{\operatorname{Im}Z_1^{\#}, f_1\}\in\mathcal{O}_{M-1}$ as $f_1=0$ and $\{\operatorname{Re}Z_1^{\#}, f_1^{\#}\}\sim 1$ I derive that $\{\operatorname{Im}Z_1^{\#}, f_1\}\in\mathcal{O}_{M-1}\cap\tilde{\mathcal{O}}_\infty$. Then
$$-i\mu^{-1}\{Z_1^{\#}, Z_1^{\#\dagger}\}=2f_1^{\#},\qquad \{Z_1^{\#}, f_1^{\#}\}\equiv 1 \mod O(\mu^{-M}) \tag{2.24}$$
and then after an appropriate $\mu^{-1}h$-FIO transformation
$$Z_1^{\#}\equiv hD_1+i\bigl(hD_3-\frac{1}{2}\mu x_1^2\bigr) \tag{2.25}$$
(again I am using a linear independence of $\nabla\operatorname{Re}Z_1^{\#}$, $\nabla\operatorname{Im}Z_1^{\#}$ and ∇f_1). However, one can replace operators $\beta(x',\mu^{-1}hD')$ by their decompositions into powers of $\mu^{-1}hD_1=\frac{1}{2}(Z_1^{\#}+Z_1^{\#\dagger})$ and $\mu^{-1}hD_3-\frac{1}{2}x_1^2=\frac{1}{2i}\mu^{-1}(Z_1^{\#}-Z_1^{\#\dagger})$ thus arriving to decomposition (2.26) below with all the operators of the $\beta(x_1,x'',\mu^{-1}hD_4;\mu^{-1}h)$ and the following statement is proven:

Proposition 2.7. *Let conditions* (2.1) *and* (2.22) *be fulfilled. Then by means of* $\mu^{-1}h$*-transform, decomposition* (2.16) *and one more* $\mu^{-1}h$*-transform one can reduce original operator* A *to the family of 3-dimensional operators*

$$\mathcal{A}_n = \frac{1}{2}\sigma\Bigl((Z_1^{\#} \bullet Z_1^{\#\dagger}) - W_n^{\#} + \sum_{2k+p+q+2m+2s\ge 3} B^{\mathrm{w}}_{kpqms} \times \bigl((2n+1)\mu h\bigr)^k Z_1^{\#\,p} Z_1^{\#\dagger\,q}\,\mu^{2-2k-2m-p-q-s}h^s\Bigr) \tag{2.26}$$

where symbol $\sigma = |\{Z_1, f_1\}|^{2/3}$ *is bounded and disjoint from 0,* $Z_1^{\#}$ *given by* (2.25) *and*

(2.27) $\mathsf{W}_n^{\#}$ *is an operator with the symbol* $\sigma^{-1}\bigl(V - f_2(2n+1)\mu h\bigr) \circ \Psi$*,* Ψ *is the smooth diffeomorphism* $\mathbb{R}^4 \to \mathbb{R}^4$*, with* $D\Psi$ *transforming* $\mathsf{Span}\bigl((0,1,0;0),(0,0,0;1)\bigr)$ *into* $\mathbb{K}_2$*,* $\mathsf{Span}\bigl((1,0,0,0),(0,0,1,0)\bigr)$ *into* $\mathbb{K}_1$ *and, in particular,* $(0,0,1;0)$ *into some element of* $\mathbb{K}_1$ *of* $(0,*,*,*)$ *form. Also* $|\det D\Psi^{-1}| = f_2$*.*

Again the first line in (2.26) is the main part.

2.2.2 Now let us consider domain where condition (2.22) is violated. Note first that $\bigl\{(x,\xi) : f_1 = Z_1 = \{Z_1, f_1\} = 0\bigr\}$ was an involutive manifold of codimension 5 in $\mathbb{R}^4 \times \mathbb{R}^4$ with

(2.28) $\nabla f_1,\ \nabla \operatorname{Re} Z_1,\ \nabla \operatorname{Im} Z_1,\ \nabla\{\operatorname{Re} Z_1, f_1\}$ and $\nabla\{\operatorname{Im} Z_1, f_1\}$ linearly independent on Λ' [13].

Also note that

$$\{Z_1, x_1\} = x_3 + ix_4, \quad \{Z_1, x_3\} = x_3 + ix_4, \quad \{Z_1, x_4\} = x_4 - ix_3 \qquad \text{as } x_1 = 0$$

and

$$\{\alpha^{\#}, \beta^{\#}\} = \{\alpha, \beta\} + i\mu^{-1}f_2^{-1}\Bigl(\{Z_2^{\dagger}, \alpha\}\{Z_2, \beta\} - \{Z_2, \alpha\}\{Z_2^{\dagger}, \beta\}\Bigr) + O(\mu^{-2}). \tag{2.29}$$

However $\{Z_2, x_j\}$ are actually arbitrary and therefore while we know that

$$\Lambda^{\#} = \bigl\{f_1^{\#} = Z_1^{\#} = \{Z_1^{\#}, f_1^{\#}\} = 0\bigr\} \tag{2.30}$$

is a manifold of codimension 5 with

(2.31) $\nabla f_1^{\#},\ \nabla \operatorname{Re} Z_1^{\#},\ \nabla \operatorname{Im} Z_1^{\#},\ \nabla\{\operatorname{Re} Z_1^{\#}, f_1\}$ and $\nabla\{\operatorname{Im} Z_1^{\#}, f_1^{\#}\}$ linearly independent on $\Lambda^{\#}$ [13),14)];

but its symplectic structure is not fixed and there is no canonical form uniformly near $\Lambda^\#$.

Instead let us consider point $\bar{z}$ ($=(0,0)\in\mathbb{R}^6$ for simplicity of notations) with

$$|f_1|\le\gamma=\epsilon r^2,\quad |\{Z_1,f_1\}|=r\qquad \text{with } r\ge\mu^{\delta-1/3}, \tag{2.32}$$

and reduce an operator to a canonical form in its $\epsilon(\gamma,r)$-vicinity with respect to $(x_1;x_2,x_3,x_4)$, leaving domain $\{|f_1|\ge\epsilon|\{Z_1,f_1\}|^2\}$ for a later.

Then after reduction of the previous subsection condition (2.32) remains valid for $Z_1^\#$ and $f_1^\#$; therefore now multiplication by a symbol $\beta=\{Z_1^\#,f_1^\#\}^\dagger\cdot|\{Z_1^\#,f_1^\#\}|^{-4/3}r^{1/3}$ provides a modified equality (2.23), namely,

$$\{Z_1^\#,f_1^\#\}=r+\alpha. \tag{2.33}$$

However, β is an uniformly smooth symbol; more precisely $\beta=\hat{\beta}(r^{-1}x',r^{-1}\xi')$ with uniformly smooth symbol $\hat{\beta}$, which means that after original rescaling $x'\mapsto rx'$ [15] $\beta=\hat{\beta}(x,\mu^{-1}hr^{-2}D')$ and $x_1^{-1}\alpha$ is of the same type.

Continuing as before[16] one can achieve a modified equality (2.24), namely,

$$-i\mu^{-1}r^{-1}\{Z_1^\#,Z_1^{\#\dagger}\}=2f_1^\#,\qquad \{Z_1^\#,f_1^\#\}\equiv r\mod O(\mu^{-s}) \tag{2.34}$$

where now $\mu^{-1}r^{-1}Z_1$ is $\mu^{-1}hr^{-2}$-PDO and then modified equality (2.25)

$$Z_1^\#\equiv\bigl(rh_*D_1+h_*D_4\bigr)+i\bigl(h_*D_3-\frac{1}{2}\mu rx_1^2\bigr) \tag{2.35}$$

where extra terms h_*D_4 and h_*D_3 appear because (original) $\nabla\operatorname{Re}Z_1$ and $\nabla\operatorname{Im}Z_1$ are linearly independent, $h_*=hr^{-1}$.

Then after another rescaling $x_1\mapsto rx_1$, $D_1\mapsto r^{-1}D_1$ (2.35) becomes

$$Z_1^\#\equiv\bigl(h_*D_1+h_*D_4\bigr)+i\bigl(h_*D_3-\frac{1}{2}\mu r^3x_1^2\bigr)$$

and after transformation $(x_1,x_3,x_4;D_1,D_3,D_4)\mapsto(x_1,x_3,x_4+x_1;D_1-D_4,D_3,D_4)$ becomes

$$Z_1^\#=h_*D_1+i\bigl(h_*D_3-\frac{1}{2}\mu_*x_1^2\bigr),\qquad h_*=hr^{-1},\quad \mu_*=\mu r^3 \tag{2.36}$$

which is exactly equality (2.25) with h and μ replaced by h_* and μ_* respectively.

[15] Since symplectic structure of $\Lambda^\#$ is not defined one cannot confine this non-smoothness to a couple of coordinates.

[16] With effective semiclassical parameter $\mu^{-1}hr^{-2}$ rather than $\mu^{-1}h$.

Meanwhile, in the original operators of type $\beta(r^{-1}x', \mu^{-1}hr^{-1}D'; \mu^{-1}hr^{-2})$ (with the smooth symbol β) were transformed subsequently into $\beta(x', \mu^{-1}hr^{-1}D'; \mu^{-1}hr^{-2})$, then into $\beta(rx_1, x'', \mu^{-1}hr^{-3}D_1, \mu^{-1}hr^{-2}D''; \mu^{-1}hr^{-2})$ and finally into

$$\beta(rx_1, x'', \mu^{-1}hr^{-3}D_1, \mu^{-1}hr^{-2}D_3, \mu^{-1}hr^{-2}D_4; \mu^{-1}hr^{-3}).$$

So, I arrive to (2.17)-like decomposition with $Z_1^\dagger$ given by (2.36) and σ, W_n and B^{w}_{kpqms} of the above type. However, one can replace such operators by their decompositions into powers of $\mu^{-1}hr^{-3}D_1 = \frac{1}{2}\mu^{-1}r^{-2}(Z_1^\# + Z_1^{\#\dagger})$ and $\mu^{-1}hr^{-2}D_3 - \frac{1}{2}r^2x_1^2 = \frac{1}{2i}\mu^{-1}r^{-1}(Z_1^\# - Z_1^{\#\dagger})$ thus arriving to decomposition (2.38) below with all the operators of the simplified type

$$\beta(rx_1, x'', \mu^{-1}hr^{-2}D_4; \mu^{-1}hr^{-3}); \tag{2.37}$$

factors $\mu^{-1}hr^{-4}$ appear because one needs to commute $Z_1^\#, Z_1^\#$ between themselves and with operators of (2.37)-type.

So, I finally arrive to

Proposition 2.8. *Let conditions* (*2.1*) *and* (*2.32*) *be fulfilled. Then by means of* $\mu^{-1}h$*-FIO transform, decomposition* (*2.14*) *and the series of rescalings and* $\mu^{-1}hr^{-2}$*-FIO transform one can reduce described above original operator* A *to the family of 3-dimensional operators*

$$\begin{aligned}\mathcal{A}_n \overset{\mathsf{def}}{=} \frac{1}{2}\sigma^\# \Bigl((Z_1^\# \bullet Z_1^{\#\dagger}) - W_n^\# + &\\ \sum_{2k+p+q+2m+2s\geq 3} B^{\mathsf{w}}_{kpqms} \times \bigl((2n+1)\mu h\bigr)^k Z_1^{\#\,p} Z_1^{\#\dagger\,q}\, \mu^{2-2k-2m-p-q-s} h^s r^{-2p-2q-4m-4s}\Bigr)&\end{aligned} \tag{2.38}$$

where $\sigma^\#$ *is transformed operator with symbol* $\sigma = |\{Z_1, f_1\}|^{2/3}r^{-2/3}$ *which is bounded and disjoint from* 0 *and* $Z_1^\#$ *given by* (*2.35*).

Again the first line in (2.38) is the main part.

Remark 2.9. (i) Here operators $\sigma^\#$, $W_n^\#$ and B^{w}_{kpqms} are of the (2.37)-type with uniformly smooth symbol β. Therefore these symbols are quantizable provided under condition (2.32) for sure;

(ii) I leave for a section 3 the the calculation of the symbol of W and the discussion of the corresponding diffeomorphism.

Figure 1: Reduction was done in the colored zone.

2.2.3 Let us look what happens with the cut-off symbol ψ as a result of these transformations. After transformations of subsubsections 2.1.1, 2.1.2 after decomposition (2.16) it becomes a matrix operator

$$\sum_{k,l} \phi_{kl}(x', \mu^{-1}hD')\mu^{-k-l} U_2^k U_2^{\dagger l}$$

with basis-shifting (in $L^2(\mathbb{R})$) operators $(2(n+1)\mu h)^{-1/2} U_2^\dagger$ and $(2n\mu h)^{-1/2} U_2$.

Then after transformations of subsubsection 2.2.2 these matrix operator becomes

$$\sum_{k,l,p,q,m,s} \phi^{\mathrm{w}}_{klpqms} U_2^k U_2^{\dagger l} Z_1^{\#\, p} Z_1^{\#\dagger\, q}\, \mu^{-k-l-m-p-q-s} h^s r^{-2p-2q-2m-4s} \tag{2.39}$$

with the scalar main part of the form (2.37).

2.3 Canonical form in the strictly outer zone

2.3.1 Now one can properly redefine the first part of the *outer zone*

$$\mathcal{Z}_{\mathsf{out},I} = \Bigl\{ C\mu^{-1/2}\mathsf{dist}(x,\Lambda)^{1/2} \le \mathsf{dist}(x,\Sigma) \le \epsilon \mathsf{dist}(x,\Lambda)^2 \Bigr\} \tag{2.40}$$

which in the notations $\mathsf{dist}(x,\Sigma) = |x_1|$ and $\mathsf{dist}(x,\Lambda) \asymp r$ is $\bigl\{\mu^{-1/2}r^{1/2} \le |x_1| \le \epsilon r^2\bigr\}$. Note that after rescaling of the previous section $x_1 \mapsto x_1 r^{-2}$, $\mu \mapsto \mu_* = \mu r^3$ it becomes $\bigl\{|x_1| \ge C\mu_*^{-1/2}\bigr\}$.

In this subsubsection I restrict myself to the part of *strictly outer zone*

$$\mathcal{Z}^*_{\mathsf{out},I} = \Bigl\{ C\mu^{3\delta-1/2}\mathsf{dist}(x,\Lambda)^{1/2} \le \mathsf{dist}(x,\Sigma) \le \epsilon \mathsf{dist}(x,\Lambda)^2 \Bigr\} \tag{2.41}$$

which in the same notations is $\bigl\{\mu^{-1/2}r^{1/2} \le |x_1| \le \epsilon r^2\bigr\}$ and after becomes $\bigl\{|x_1| \ge C\mu_*^{3\delta-1/2}\bigr\}$ (may be with a different $\delta > 0$) and provides a nice quantization of all symbols below.

Then since $\mu_* \ge h^{-\delta}$ one can apply the canonical form of the Schrödinger operator with the strong magnetic field. Let us consider an operator obtained after reduction of the previous subsection; there $|x_1| \asymp \gamma_* = \gamma r^{-2}$ if originally $|x_1| \asymp \gamma$. Notation $\gamma = r^2$ was temporary.

Applying rescaling $(x_1, x_3) \mapsto (x_1\gamma_*, x_3\gamma_*)$ one arrives to transformed (2.36) of the same type but with h_* and μ_* replaced by $h_*\gamma_*^{-1} = hr\gamma^{-1}$ and $\mu_*\gamma_*^2 = \mu\gamma^2 r^{-1}$ respectively:

$$Z_1^{\#} = \mu\gamma^2 r^{-1}\Bigl(\hbar D_1 + i\bigl(\hbar D_3 + \frac{1}{4}x_1^2\bigr)\Bigr) \tag{2.42}$$

with $|x_1|\asymp 1$ and

$$\hbar=\mu^{-1}hr^2\gamma^{-3}\le c\mu^{1/2}h^{1+3\delta}r^{1/2}\ll ch^{1/2}. \tag{2.43}$$

Also $\mu^{-1}hr^{-2}D_3\mapsto \mu^{-1}h\gamma^{-1}D_3$.

Let us consider the part where $x_1>0$; another part is separated and treated exactly in the same manner. Introducing

$$U_1=x_1^{-1/2}\Bigl(\hbar D_1+i\bigl(\hbar D_3+\frac{1}{4}x_1^2\bigr)\Bigr) \tag{2.44}$$

one can see easily that

$$|U_1|\le c\nu^{-1},\qquad \nu\overset{\mathsf{def}}{=}\mu\gamma^2r^{-1}\ge h^{-\delta}, \tag{2.45}$$

$$\{U_1,U_1^\dagger\}\equiv 1\qquad \operatorname{mod}\ U_1,U_1^\dagger \tag{2.46}$$

where (2.45) follows from $|Z_1^\#|\le c$ and (2.46) is understood in the sense of $\hbar$-symbols.

Therefore there exists $\hbar$-FIO transformation $\mathcal{F}'$ of $(x_1,x_3;\hbar D_1,\hbar D_3)$ thus not affecting $(x_4,\mu^{-1}hr^{-2}D_4)$ so that

$$U_1\mapsto U_1^\#\equiv \hbar D_1+ix_1\qquad \operatorname{mod}\ \mathcal{O}_2(U_1,U_1^\dagger); \tag{2.47}$$

then operator (2.38) is transformed into

$$\frac{1}{2}\sigma\sigma_1\Bigl(\nu^2U_1^\#\bullet U_1^{\#\dagger}-W_n^{\#\#}+ \sum_{2k+p+q+2m+2s+2l\ge 3} B^{\mathsf{w}}_{kpqmsl}\times\bigl((2n+1)\mu h\bigr)^k\bigl(\nu U_1^\#\bigr)^p\bigl(\nu U_1^{\#\dagger}\bigr)^q\mu^{2-2k-2m-p-q-s}h^sr^{-2p-2q-4m-4s}\hbar^l\Bigr) \tag{2.48}$$

with $W_n^{\#\#}$ the result of transformation of $x_1^{-1}W_n^\#$.

Here all the operators σ, σ_1, $W_n^{\#\#}$ and B^{w}_{kpqmsl} of the type

$$\beta(x_1,x_3,x_4;\hbar D_1,\hbar D_3,\mu^{-1}hr^{-2}D_4).$$

Note that $\hbar\nu=hr\gamma^{-1}\ll\mu^{1/2}hr^{1/2}\le h^{1/2}$ in the zone in question. One can decompose all such operators into power series with respect to x_1, $\hbar D_1$ and rewrite (2.48) with all operators of the type

$$\beta(x_3,x_4;\hbar D_3,\mu^{-1}hr^{-2}D_4),\qquad \hbar=\mu^{-1}hr^2\gamma^{-3} \tag{2.49}$$

and with $\bigl(\nu U_1^{\#}\bigr)^p$, $\bigl(\nu U_1^{\#\dagger}\bigr)^q$ replaced by $\bigl(\nu U_1^{\#}\bigr)^{p+p'}\nu^{-p'}$, $\bigl(\nu U_1^{\#\dagger}\bigr)^{q+q'}\nu^{-q'}$ respectively since $\nu \ge h^{-\delta}$.

Repeating arguments of the previous subsections one can transform this operator into

$$\begin{aligned}\text{(2.50)}\quad &\frac{1}{2}\sigma\sigma_1\Bigl(\nu^2 U_1^{\#}\bullet U_1^{\#\dagger} - W_n^{\#\#}+\\ &\sum_{2q+2l+2j\ge 3} B^{\mathrm{w}}_{qjl}\times\bigl(\nu^2 U_1^{\#}\bullet U_1^{\#\dagger}\bigr)^q\,\nu^{2-2q-2j}\hbar^l+\\ &\sum_{2k+2q+2m+2s+2l\ge 3} B^{\mathrm{w}}_{kqjmsl}\times\bigl((2n+1)\mu h\bigr)^k\bigl(\nu^2 U_1^{\#}\bullet U_1^{\#\dagger}\bigr)^q\,\mu^{2-2k-2m-2q-s}h^s r^{-4q-4m-4s}\nu^{-2j}\hbar^l\Bigr)\end{aligned}$$

where the second line comes from decomposing of $W_n^{\#\#}$ and because it is rescaled before

$$|B_{qjl}|\le C\gamma r^{-1}, \tag{2.51}$$

Now one can apply decomposition with respect to $\hbar^{-1/4}\upsilon_p(\hbar^{-1/2}x_1)$:

$$u_{nn'}(x',y',t)=\sum_{p,p'\in\mathbb{Z}^+}\hbar^{-1}u_{npn'p'}(x'',y'',t)\,\upsilon_p\bigl(\hbar^{-1/2}x_1\bigr)\upsilon_{p'}\bigl(\hbar^{-1/2}y_1\bigr) \tag{2.52}$$

with $x''=(x_3,x_4)$. Then the operator (2.50) becomes

$$\begin{aligned}\text{(2.53)}\quad &\frac{1}{2}\sigma\sigma_1\Bigl((2p+1)\mu h\gamma - W_n^{\#\#}+\\ &\sum_{2q+2l+2j\ge 3} B^{\mathrm{w}}_{qjl}\times\bigl((2p+1)\mu h\gamma\bigr)^q\,\nu^{2-2q-2j}\hbar^l+\\ &\sum_{2k+2q+2m+2s+2l\ge 3} B^{\mathrm{w}}_{kqjmsl}\times\bigl((2n+1)\mu h\bigr)^k\bigl((2p+1)\mu h\gamma\bigr)^q\,\mu^{2-2k-2m-2q-s}h^s r^{-4q-4m-4s}\nu^{-2j}\hbar^l\Bigr).\end{aligned}$$

since $\nu^2\hbar=\mu h\gamma$ and $2k+2q+2m+2s+2j+2l\ge 3$ is equivalent to $k+q+m+s+j+l\ge 2$.

In the decomposition (2.53) the role of m, s, j and l is just to bound a magnitude and indicate dependence on μ and h since dependence on r,γ (but not magnitude) is not important. However $\nu=\mu\cdot\gamma^2 r^{-1}\le\mu r^2$, $\hbar=\mu^{-1}h\cdot r^2\gamma^{-3}\ge\mu^{-1}hr^{-4}$; therefore one can join terms with $s>0$ or $m>0$ to the terms with $s=0$ and $m=0$ without changing $s+l$, $m+j$ thus getting the simplified correction term

$$\sum_{k+q+j+l\ge 2} B^{\mathrm{w}}_{kqjmsl}\times\bigl((2n+1)\mu h\bigr)^k\bigl((2p+1)\mu h\gamma\bigr)^q\,\mu^{2-2k-2q}h^s r^{-4q}\nu^{-2j}\hbar^l$$

Therefore I arrive to the following

Proposition 2.10. *Let condition (2.1) be fulfilled. Let us consider zone $\mathcal{Z}^*_{\mathsf{out},I}$, described by (2.40).*

Then by means of $\mu^{-1}hr^{-2}$-FIO transforms, decomposition (2.16), the "special" $\hbar$-FIO transform and decomposition (2.52) one can reduce the original operator A to the family of 2-dimensional operators

$$\mathcal{A}_{np} \overset{\mathsf{def}}{=} \frac{1}{2}\sigma\sigma_1\Bigl(H_{pn}+ \sum_{k+q+j+l\geq 2} B^{\mathsf{w}}_{kqjmsl}\times\bigl((2n+1)\mu h\bigr)^k\bigl((2p+1)\mu h\gamma\bigr)^q\mu^{2-2k-2q}h^s r^{-4q}\nu^{-2j}\hbar^l\Bigr) \tag{2.54}$$

with $n\leq C_0/(\mu h)$, $p\leq C_0/(\mu h\gamma)$ where all operators are of (2.49)-type and σ, σ_1 are bounded and disjoint from 0.

Remark 2.11. (i) While in the canonical form (2.38) only one cyclotron movement was separated, in the canonical form (2.54) both of them are separated leaving only pure drift movement which will be studied in the next section;
(ii) I leave for a section 3 the discussion of the actual elements where reduction was done, corresponding diffeomorphism and the calculation of the symbol of W;
(iii) Cut-off symbol ψ which after previous transformations became (2.39) now becomes

$$\sum_{k,i,p,q,s,l}\phi_{kipqsl}\times U_2^i U_2^{\dagger\,k} U_1^{\#\,p} U_1^{\#\dagger\,q}\mu^{-i-k-2m-p-q-s}h^s r^{-2p-2q-4m-4s}\nu^{-2j-i-k}\hbar^l \tag{2.55}$$

with (2.49)-type operators ϕ_{kipqsl}.

2.3.2 Now my goal is to achieve a similar reduction in the second part of the *outer zone*

$$\mathcal{Z}^*_{\mathsf{out},II}=\Bigl\{C\max\bigl(\mathsf{dist}(x,\Lambda)^2,\mu^{4\delta-2/3}\bigr)\leq\mathsf{dist}(x,\Sigma)\leq\epsilon\Bigr\}. \tag{2.56}$$

Then the previous subsection 2.2 becomes irrelevant; however I can start from (2.17). Again with no loss of the generality one can assume that $f_1^{\#}=x_1$. Let us consider a point $\bar{z}=(\bar{x}',\bar{\xi}')\in\mathbb{R}^6$ and an element (2.55) with $\gamma=|\bar{x}_1'|$ and $r=\gamma^{1/2}$ (exactly as in the previous subsection it was defined the other way. Without any loss of the generality one can assume that $\bar{z}=(\gamma,0,\dots,0)$.

Consider Taylor decomposition (with a remainder) of $\mu^{-1}Z_1^{\#}$ with respect to x', $\mu^{-1}hD'$ and r (in r-vicinity of 0 all of them are $O(r)$); then due to $\{Z_1^{\#},Z_1^{\#\dagger}\}=O(r)$, $\{Z_1,x_1\}=O(r)$ with no loss of the generality one can assume that $\mu^{-1}Z_1^{\#}=\mu^{-1}hD_3+i\mu^{-1}hD_4+$

$O(r^2)$. Note that $\mu^{-1}hD_1$ could be either as square (or higher degree) or with a linear or quadratic factor with respect to other variables. Also note that the quadratic terms in $O(r^2)$ containing exclusively x_3, x_4 must commute with D_3+iD_4 (otherwise $\{Z_1^{\#}, Z_1^{\#\dagger}\}\sim 2x_1$ would not be possible) and therefore must be of the form $c(x_3+ix_4)^2$. However one could remove such terms by $\mu^{-1}h$-PDO affecting only $x'', \mu^{-1}hD''$ and leaving x_1, $\mu^{-1}hD_1$ intact. So, terms in the decomposition must contain either one of the factors x_1, $\mu^{-1}hD_3$, $\mu^{-1}hD_4$, or two factors $\mu^{-1}hD_1$, or three factors x_3, x_4; also x_1 must be accompanied by $O(r)$-type factor.

Then scaling $x''\mapsto rx'$, $x_1\mapsto r^2x_1$, $D''\mapsto r^{-1}D''$, $D_1\mapsto r^{-2}D_1$ and dividing by r^3 one gets completely legitimate $\mu^{-1}hr^{-4}$-PDO where each factor $\mu^{-1}hD''$ can accommodate division by r^3 and $\mu^{-1}hD_1$ can accommodate division by r^2 but there either at least two such factors, or there is an extra $O(r)$ type factor. This operator, with the semiclassical parameter $\hbar=\mu^{-1}h\gamma^{-2}$ satisfies again $\{Z_1^{\#}, Z_1^{\#\dagger}\}\sim 2x_1$ but must be considered near point $(1,0,\dots,0)$ and then by the standard arguments one can reduce operator to the canonical form (2.54) with all the operators of the type

$$\beta(x_3,x_4;\hbar D_3,\hbar D_4),\qquad \hbar=\mu^{-1}h\gamma^{-2}. \tag{2.57}$$

Note that as $r=\gamma^{1/2}$ this is consistent with the definition of $\hbar$ in the zone $\mathcal{Z}_{\mathsf{out},I}$ but form (2.57) is more general then (2.49).

So, I arrive to

Proposition 2.12. *Let condition (2.1) be fulfilled. Then one can reduce the original operator A to the family (2.54) of 2-dimensional operators with $n\le C_0/(\mu h)$, $p\le C_0/(\mu h\gamma)$ where all operators are of (2.49)-type and σ, σ_1 are bounded and disjoint from 0.*

Remark 2.13. (i) All statements (i)-(iii) of remark 2.11 remain valid.

2.3.3 I would like to finish this section by the following

Remark 2.14. In $2D$-case canonical form would be [Ivr6]

$$\mathcal{A}_p=\sigma\sigma_1\Bigl((2p+1)\mu h\gamma - W^{\#\#}+ \sum_{2l+2j+2t\ge 3} B^{\mathsf{w}}_{ljt}\times\bigl((2p+1)\mu h\gamma\bigr)^l\times \mu^{2-2l-2j-t}h^t\gamma^{-4j-3t}\Bigr) \tag{2.58}$$

with

$$W_n^{\#\#}=\beta(\gamma^{-1}x_3,\mu^{-1}h\gamma^{-2}D_3). \tag{2.59}$$

There instead of 2-parameter family of 2-dimensional PDOs I had only 1-parametric family of 1-dimensional PDOs.

3 Estimates

In this section I derive remainder estimates with the main part in the standard but rather implicit form which is the sum of expressions (0.12)

$$h^{-1}\sum_\iota \int_{-\infty}^0 \Bigl(F_{t\to h^{-1}\tau}\bar\chi_{T_\iota}(t)\Gamma u Q_{\iota y}^t\Bigr)\,d\tau \tag{3.1}$$

with Q_ι partition of unity and an appropriate T_ι where as in my previous papers $\bar\chi_T(t)=\bar\chi(t/T)$ and $\bar\chi$ is supported in $[-1,1]$ and equal 1 at $[-\frac{1}{2},\frac{1}{2}]$, while the remainder estimate could depend on non-degeneracy condition.

I am going to consider different zones and apply different approaches and use different canonical forms in each of them; I start from the strictly outer zone.

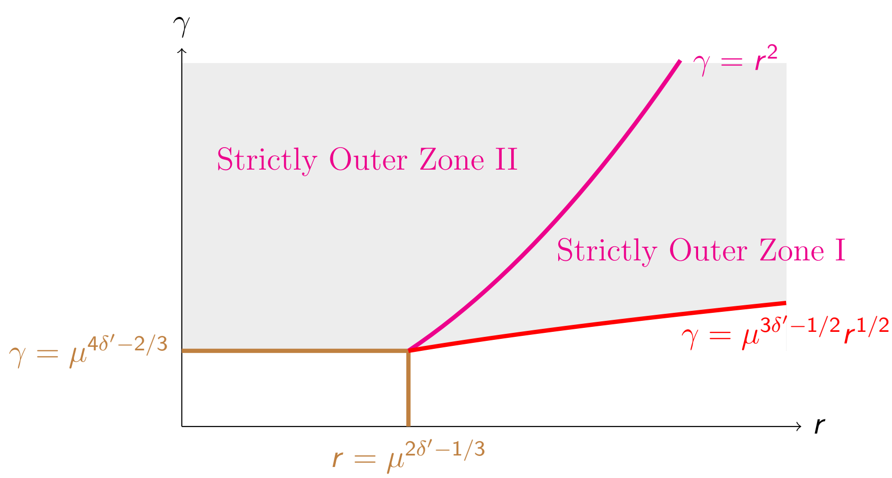

Figure 2: My first target is strictly outer zone.

3.1 Estimates in the strictly outer zone. I

I will start from the most massive *Strictly Outer Zone I* $\mathcal{Z}^*_{\mathsf{out},I}$ described in (2.40) with an arbitrarily small exponent $\delta_0>0$ and sufficiently small constant $\epsilon>0$.

3.1.1 After rescalings and transformations described in the previous section the original operator *somewhere* there was reduced to the family of two-dimensional $(\hbar_3,\hbar_4)$-PDOs

$\mathcal{A}_{pn}(x'', \hbar_3 D_3, \hbar_4 D_4)$ given by (2.54) with

$$\hbar_3 = \mu^{-1} h r^2 \gamma^{-3} \ll \mu^{1/2} h, \qquad \hbar_4 = \mu^{-1} h r^{-2} \ll \mu^{-1/3} h; \tag{3.2}$$

one needs to consider only magnetic numbers

$$n \le C_0(\mu h)^{-1}, \qquad p \le C_0(\mu h\gamma)^{-1} \qquad \text{as} \ \ \mathsf{dist}(x, \Sigma) \asymp \gamma; \tag{3.3}$$

these two restrictions mean that energy levels do not exceed ϵ.

These final operators are defined in $B(0,1)$. Therefore

(3.4) A contribution of each $(\mathsf{pair}, \mathsf{element})$[17] to to the asymptotics is $O\bigl(\hbar_3^{-1}\hbar_4^{-1}\bigr) = O\bigl(\mu^2 h^{-2}\gamma^3\bigr)$[18].

However, one can see easily that this element before the final reduction was the "slab" $B(0,1) \cap \{|x_1 - \bar{x}_1| + |x_3 - \bar{x}_3| \le \epsilon\gamma_*\}$ with $\gamma_* = \gamma r^{-2}$, $|\bar{x}_1| \asymp \gamma_*$ where here $B(0,1)$ is an image of the original element $B(y', r) \cap \mathcal{Z}_{\mathsf{out},I}$. Therefore the total contribution of the given pair (p, n) over subzone

$$\mathcal{Z}_{(r,\gamma)} = \bigl\{\gamma \le \mathsf{dist}(x, \Sigma) \le 2\gamma, r \le \mathsf{dist}(x, \Lambda) \le 2r\bigr\}$$

with $r \ge \epsilon\gamma^{1/2}$ to the asymptotics is $O\bigl(\mu^2 h^{-2}\gamma^3 \times r^2\gamma^{-1}\bigr) = O\bigl(\mu^2 h^{-2}\gamma^2 r^2\bigr)$ and the total contribution of subzone $\mathcal{Z}_{(r,\gamma)}$ to the asymptotics is $O\bigl(\mu^2 h^{-2}\gamma^2 r^2 \times (\mu h\gamma)^{-1} \times (\mu h)^{-1}\bigr) = O\bigl(h^{-4}\gamma r^2\bigr)$ where the second and the third factors are numbers of permitted indices p and n respectively.

Then the total contribution of zone $\mathcal{Z}^*_{\mathsf{out},I}$ to asymptotics is $O\bigl(h^{-4}\bigr)$[19].

The same approach works in $\mathcal{Z}^*_{\mathsf{out},II}$: here one has the family of two-dimensional $\hbar_3$-PDO $\mathcal{A}_{pn}(x'', \hbar_3 D_3, \hbar_3 D_4)$ with $\hbar_3 = \mu^{-1} h\gamma^{-2}$. Therefore a contribution of each $(\mathsf{pair}, \mathsf{element})$ to to the asymptotics is $O(\hbar_3^{-2}) = O(\mu^2 h^{-2}\gamma^4)$. However there was nondiscriminatory rescaling $(x_3, x_4, \xi_3, \xi_4) \mapsto (r^{-1}x_3, r^{-1}x_4, r^{-1}\xi_3, r^{-1}\xi_4)$ while the number of such balls should be $O(r^{-2})$ since $\mathsf{codim}_\Sigma \Lambda = 2$.

Therefore the total contribution of the given pair (p, n) over subzone

$$\bar{\mathcal{Z}}_{(r,\gamma)} = \bigl\{\gamma \le \mathsf{dist}(x, \Sigma) \le 2\gamma, \mathsf{dist}(x, \Lambda) \le 2r\bigr\}$$

with $r = \gamma^{1/2}$ to the asymptotics is $O(\mu^2 h^{-2}\gamma^3)$, the total contribution of subzone $\bar{\mathcal{Z}}_{(r,\gamma)}$ to the asymptotics is $O(h^{-4}\gamma^2)$ and the total contribution of zone $\mathcal{Z}^*_{\mathsf{out},I}$ to asymptotics is $O(h^{-4})$.

Now I need to analyze the contribution of such elements to the remainder estimate.

[17] Where here and below a pair is (p, n) and an element means the final element $B(0,1)$.

[18] And for many pairs it is actually of this magnitude.

[19] And is actually of this magnitude under condition (0.2) as $\mu \le \epsilon h^{-1}$. These results are consistent with what one can get without reduction.

3.1.2 Let us consider some index pair (p,n) and some final $B(0,1)$ element. Assume first that after all these transformations operator $\mathcal{A}_{pn}$ is *regular* on this $(\mathsf{pair,element})$ in the sense of (3.5),(3.6) and (3.8)

$$\eta_\alpha |\nabla^\alpha_{x'',\xi''}\mathcal{A}_{pn}| \le c_\alpha \ell \qquad \forall \alpha : 0<|\alpha|\le K, \tag{3.5}$$

with $\eta_\alpha = 1$ as $\alpha=(*,0,*,0)$ and $\eta_\alpha = \gamma r^{-2}$ otherwise and ℓ satisfying

$$\ell \ge h^{-K_0}. \tag{3.6}$$

Then the propagation speed with respect to $(x_3,\hbar_3 D_3)$ is $O(\ell\hbar_3/h)=O(\ell\mu^{-1}r^2\gamma^{-3})$ and the propagation speed with respect to $(x_4,\hbar_4 D_4)$ is $O(\ell\hbar_4/h\times r^2/\gamma)=O(\ell\hbar_3/h)$ as well. Therefore the dynamics starting in $B(0,\frac{1}{2})$ is retained in $B(0,1)$ as $|t|\le T_1$ with

$$T_1=\epsilon h\ell^{-1}\hbar_3^{-1}\asymp \mu\gamma^3 r^{-2}\ell^{-1}; \tag{3.7}$$

I remind that $\hbar_3\gg\hbar_4$. In the same time under respective assumptions

$$|\nabla_{x_3,\xi_3}\mathcal{A}_{pn}| + \zeta|\nabla_{x_4,\xi_4}\mathcal{A}_{pn}| \asymp \ell, \quad \zeta=\gamma r^{-2} \tag{3.8}$$

one can pick up T_0 from the uncertainty principle[20] $\ell\hbar_j h^{-1}T_0\ge \hbar_j h^{-\delta'}$ [21] or equivalently

$$T_0 = Ch^{1-\delta'}\ell^{-1}. \tag{3.9}$$

with an arbitrarily small exponent $\delta'>0$.

Then the contribution of this $(\mathsf{pair,element})$ to the remainder estimate does not exceed

$$C\hbar_3^{-1}\hbar_4^{-1}\times T_0T_1^{-1}\asymp \hbar_4^{-1}h^{-\delta'}=\mu h^{-1-\delta'}r^2 \tag{3.10}$$

where $C\hbar_3^{-1}\hbar_4^{-1}\times T_0$ estimates $\sup_{|\tau|\le\epsilon}|F_{t\to h^{-1}\tau}\bar\chi_T(t)\Gamma\bigl(u\psi\bigr)|$ and factor T_1^{-1} is due to the standard Tauberian approach.

So, in comparison with the estimate of the contribution to the asymptotics (3.4) the estimate of the contribution to the asymptotics picked an extra factor $\hbar_3 h^{-\delta'}$ with an arbitrarily small exponent $\delta'>0$; actually one can reduce δ' to 0 but it is not needed here.

Let us pick up $(p,\mathsf{element})$, then break this element into ζ-subelements with respect to (x_4,ξ_4); then on each such subelement operator $\mathcal{A}_{pn}$ is elliptic[22] $|\mathcal{A}_{pn}|\ge\epsilon\ell$ for all indices n

[20] I don't need the logarithmic uncertainty principle in this place and use the simpler version of the microlocal uncertainty principle.

[21] Alternatively one could scale $x_4\mapsto(\hbar_4/\hbar_3)^{1/2}x_4$, $\xi_4\mapsto(\hbar_4/\hbar_3)^{1/2}\xi_1$, $\hbar_4\mapsto\hbar_3$.

[22] See next subsubsection for justification; to prove (3.14) and (3.15) which are main results of this subsubsection one does not need these arguments.

but $C\bigl(\ell/(\mu h)+1\bigr)$. Therefore the total contribution to the remainder the pair $(p,\mathsf{element})$ combination such that conditions (3.5),(3.6) and (3.8) are fulfilled with $\ell\le\bar\ell$ does not exceed

$$C\mu h^{-1-\delta'}r^2\times\bigl(\bar\ell(\mu h)^{-1}+1\bigr)=C\bigl(\bar\ell+\mu h\bigr)h^{-2-\delta'}r^2. \tag{3.11}$$

Then summation over $\mathcal{Z}_{(r,\gamma)}$ results in

$$Ch^{-4}\gamma r^2\times\mu^{-1}hr^2\gamma^{-3}\times h^{-\delta'}(\bar\ell+\mu h)=C\bigl(\bar\ell+\mu h\bigr)\mu^{-1}h^{-3-\delta'}r^4\gamma^{-2} \tag{3.12}$$

and therefore

(3.13) The total contribution to the remainder of all $(\mathsf{pair},\mathsf{element})$ combinations residing in $\mathcal{Z}_{(r,\gamma)}\subset\mathcal{Z}^*_{\mathsf{out},I}$ and satisfying conditions (3.5),(3.6) and (3.8) with $\ell\le\bar\ell$ does not exceed (3.12).

It implies

(3.14) The total contribution to the remainder of all $(\mathsf{pair},\mathsf{element})$ combinations residing in the *far outer zone* $\mathcal{Z}^*_{\mathsf{out},I}\cap\bigl\{\mathsf{dist}(x,\Sigma)\ge\mu^{-1/4}h^{-\delta'}\mathsf{dist}(x,\Lambda)^2\bigr\}$ and satisfying conditions (3.5),(3.6) and (3.8) does not exceed $C\mu^{-1/2}h^{-3}$.

The same arguments work for $\bar{\bar{\mathcal{Z}}}_{(r,\gamma)}$ with $r=\gamma^{1/2}$ resulting in the estimate $C\mu^{-1}h^{-3-\delta'}$ of its contribution to the remainder; summation over $\mathcal{Z}^*_{\mathsf{out},II}$ results in expression $o(\mu^{-1/2}h^{-3})$. Therefore

(3.15) The total contribution to the remainder of all $(\mathsf{pair},\mathsf{element})$ combinations residing in $\mathcal{Z}^*_{\mathsf{out},II}$ and satisfying conditions (3.5),(3.6) and (3.8) with $\zeta=1$[23] does not exceed $C\mu^{-1/2}h^{-3}$.

I consider the exceptional regular elements (not covered by (3.14) in the next subsubsection. The rest of the analysis of this subsection is devoted to detecting and analysis of the irregular elements.

[23] One should pick up here $\zeta_1=1$ because $\hbar_3=\hbar_4$.

3.1.3 Note that

$$H_{pn} = \Bigl(\phi_1^{-1}\bigl(V-(2n+1)\mu h f_2-(2p+1)\mu h f_1\bigr)\Bigr)\circ\Psi \tag{3.16}$$

with uniformly smooth map Ψ which does not depend on V; this exonerates elliptic arguments of the previous subsubsection and in particular (3.13) since perturbation part satisfies $|\partial_n \mathcal{B}_{pn}|\ll \mu h$; one can consider n and p as continuous parameters.

Proposition 3.1. *(i) After reduction in $\mathcal{Z}^*_{\mathsf{out},I}$ symbols $W\circ\Psi$, $f_1\circ\Psi$, $f_2\circ\Psi$ satisfy conditions $(3.5),(3.6)$ with $\ell=\gamma r^{-1}$ and indicated above η_α and ζ; furthermore*

$$|\nabla_{x_3,\xi_3} f_1\circ\Psi|\asymp\gamma; \tag{3.17}$$

(ii) After reduction in $\mathcal{Z}^*_{\mathsf{out},II}$ symbols $W\circ\Psi$, $f_1\circ\Psi$, $f_2\circ\Psi$ satisfy conditions $(3.5),(3.6)$ with $\ell=\gamma r^{-1}$; furthermore

$$|\nabla_{x'',\xi''} f_1\circ\Psi|\asymp\gamma. \tag{3.18}$$

Proof. Proof follows easily from the reductions. More precise results will be needed and proven later. □

Therefore

(3.19) As $p\ge C_0/(\mu h r)$ symbols H_{pq} satisfy conditions (3.5),(3.6) and (3.8 with $\ell=\mu h\gamma p\ge C_0\gamma/r$.

On the other hand one can prove easily from the reduction that

(3.20) Perturbation symbols $\mathcal{B}_{pn}$ satisfy conditions (3.5) and (3.6) with ℓ replaced by $\ell' = h^2\gamma^{-2}r^2(p+1)^2+\mu^{-1}h\gamma^{-3}r^2\ll \mu h\gamma(p+1)$.

Note that if I restrict myself only by $p\le C_0/(\mu h r)$ (condition to be explained later) instead of $p\le C_0/(\mu h\gamma)$, transition from (3.11) to (3.12) picks up an extra factor γ/r and therefore

(3.21) The total contribution to the remainder of all $(\mathsf{pair},\mathsf{element})$ combinations with $p\le C_0/(\mu h r)$, residing in $\mathcal{Z}^*_{\mathsf{out},I}$ and satisfying conditions (3.5),(3.6) and (3.8) does not exceed $C\mu^{-1/2}h^{-3}$.

Now I want to cover indices $p \geq C_0/(\mu hr)$ and elements in $\mathcal{Z}^*_{\mathsf{out},l}$ which are not in the far outer zone and prove that their total contribution to the remainder also does not exceed $C\mu^{-1/2}h^{-3}$. To achieve this I plan gain a factor γr^{-2} in the estimates of the previous subsubsection, starting from (3.10) by increasing T_1 given by (3.7) to

$$T_2 = \epsilon r\mu_* \gamma_*^2 = \epsilon\mu\gamma^2. \tag{3.22}$$

Note that (3.7) appeared because it is a time for which propagation is confined to $B(0,1)$ in the final reduction, for larger t shift with respect to (x_3, ξ_3) would be too large while shift with respect to (x_4, ξ_4) would be still less than ϵ as $|t| \leq T_2$.

However considering propagation after intermediate reduction (i.e. reduction of subsection 2.2) the shift with respect to x_3 would be of magnitude $\mu_*^{-1}\gamma_*^{-2}(\mu h\gamma p)|t|$ which would be observable but less than 1 as $T_1 \leq |t| \leq T_2$; this propagation just moves to the adjacent "slabs" in the intermediate element $B(0,1)$ corresponding to the different final elements.

Then the contribution of such $(\mathsf{pair},\mathsf{element})$ to the remainder estimate is given by a modified expression (3.10)

$$C\hbar_3^{-1}\hbar_4^{-1} \times T_0 T_2^{-1} \asymp \mu h^{-1-\delta'}\gamma\ell^{-1} \asymp \mu h^{-1-\delta'}\gamma(\mu h\gamma p)^{-1} \tag{3.23}$$

which leads to modified expression (3.12)

$$\mu h^{-1-\delta'}\gamma r^{-2} \times (\mu h)^{-1} \times (\mu h\gamma)^{-1}|\log h| \times r^2\gamma^{-1} \tag{3.24}$$

where the second, the third and the fourth factors estimate respectively the number of indices n, the sum of $(\mu h\gamma p)^{-1}$ with respect to p and the number of the final elements with given (r,γ).

Then the sum with respect to γ could be estimated by by the same expression with $\gamma = \mu^{-1/2-\delta}r^{1/2}$, which is $C\mu^{-1/2-\delta}h^{-3-\delta'}r^{3/2}|\log h|$; finally summation with respect to r results in $O(\mu^{-1/2}h^{-3})$ as $\delta' > 0$ is small enough..

Therefore I arrive to

(3.25) The total contribution to the remainder of all $(\mathsf{pair},\mathsf{element})$ combinations with $p \geq C_0/(\mu hr)$, residing in $\mathcal{Z}^*_{\mathsf{out},l}$ does not exceed $C\mu^{-1/2}h^{-3}$.

which together with (3.21 concludes the analysis of regular elements; their total contribution to the remainder is $O(\mu^{-1/2}h^{-3})$.

3.1.4 Therefore in what follows I am interested only in non-regular elements, with $p \le C_0/(\mu h r)$; then H_{pn} with n violating ellipticity condition satisfies (3.5),(3.6) with $\ell = \gamma r^{-1}$ in $\mathcal{Z}^*_{\mathsf{out},I}$ and with $\ell = \gamma r^{-1}$, $\eta_\alpha = (1,1,1,1)$ in $\mathcal{Z}^*_{\mathsf{out},II}$.

First consider elements in $\mathcal{Z}^*_{\mathsf{out},I}$ with fixed $p \le C_0/(\mu h r)$ and n. After rescaling

$$x_4 \mapsto \gamma r^{-2} x_4, \quad \xi_4 \mapsto \gamma r^{-2}\xi_4 \quad \hbar_4 \mapsto \hbar'_4 \overset{\mathsf{def}}{=} \mu^{-1} h \gamma^{-2} r^2 \tag{3.26}$$

symbol $\gamma^{-1} r H_{pn}$ becomes uniformly smooth and therefore one can introduce a scaling function

$$\ell = \ell_{pn}(x'', \xi'') = \epsilon r \gamma^{-1} |\nabla_{x'',\xi''} \mathcal{A}_{pn}| + \bar\ell_0, \qquad \bar\ell_0 = C\hbar_3^{1/2} h^{-\delta''}; \tag{3.27}$$

obviously $|\nabla \ell| \le \frac{1}{2}$. Let us introduce ℓ-admissible partition of $B(0,1)$. Then the symbols are still quantizable after rescaling $(x'',\xi'') \mapsto \ell^{-1}(x'',\xi'')$, $\hbar_3 \mapsto \ell^{-2}\hbar_3$, $\hbar'_4 \mapsto \ell^{-2}\hbar'_4$.

Further, as $\ell \ge 2\bar\ell_0$, the propagation speed does not exceed

$$v \asymp \mu^{-1}\gamma^{-3} r^2 |\nabla \mathcal{A}_{pn}| \asymp \mu^{-1} r \gamma^{-2} \ell \tag{3.28}$$

in the corresponding partition subelement[24] (in the coordinates before rescaling of the subelements) and therefore the dynamics started in $B(z,\ell(z))$ remains in $B(z, 2\ell(z))$ as $|t| \le T_1$ with

$$T_1 = \epsilon \ell v^{-1} = \epsilon \mu r^{-1} \gamma^2. \tag{3.29}$$

On the other hand, the shift $v|t|$ is observable as $|t| \ge T_0$ with T_0 defined from the uncertainty principle

$$v T_0 \times \ell \ge \hbar_3 h^{-\delta'}. \tag{3.30}$$

Really, if $|\nabla_{x_3,\xi_3} \mathcal{A}_{pn}| \asymp \ell\gamma$ then the propagation speed with respect to (x_3,ξ_3) is exactly of magnitude v and the uncertainty principle is due to (3.30). Otherwise $|\nabla_{x_4,\xi_4}\mathcal{A}_{pn}| \asymp \ell\gamma$ and the propagation speed with respect to (x_4,ξ_4) is exactly of magnitude $v\gamma$ and the uncertainty principle is $\gamma v T_0 \times \ell \ge \hbar_2 h^{-\delta'} r^{-1}$ which is due to (3.30) as well.

So one can pick up

$$T_0 = \hbar_3 h^{-\delta'} v^{-1} \ell^{-1} \asymp h^{1-\delta'} \gamma^{-1} r \ell^{-2}. \tag{3.31}$$

Therefore the contribution of this partition subelement to the remainder does not exceed

$$C\hbar_3^{-1} \hbar_4'^{-1} \ell^4 \times T_0 T_1^{-1} = C\mu h^{-1-\delta'} \gamma^2 r^{-2} \ell^2 \tag{3.32}$$

[24] I call this partition elements "subelements" because actually $B(0,1)$ itself is a rescaled partition element by itself.

while in comparison its contribution to the main part of asymptotics does not exceed $C\hbar_3^{-1}\hbar_4'^{-1}\ell^4 = C\mu^2 h^{-2}\gamma^5\ell^4$.

On the other hand, the contributions of the subelement with $\ell \le \bar{\ell}_0$ to both the main part and the remainder do not exceed

$$C\hbar_3^{-1}\hbar_4'^{-1}\ell^4 = C\mu^2 h^{-2} r^{-4}\gamma^5\bar{\ell}_0^4 \tag{3.33}$$

which is less than (3.32) (with a larger but still arbitrarily small exponent δ').

Let us notice that for given index p and ℓ-element operator $\mathcal{A}_{pn}$ is elliptic as $|n-\bar{n}|\mu h \ge C_0\ell^2\gamma r^{-1}$ for some $\bar{n}$; therefore ellipticity fails for no more than $\bigl(C_0\ell^2\gamma r^{-1}(\mu h)^{-1}+1\bigr)$ indices n.

Then the contribution to the remainder of all the indices n (while index p and ℓ-subelement remain fixed) does not exceed

$$C\mu h^{-1-\delta'}\gamma^2\ell^2 \times \bigl(C_0\ell^2\gamma(\mu h r)^{-1}+1\bigr). \tag{3.34}$$

As

$$\ell \ge \bar{\ell}_1 \overset{\mathsf{def}}{=} \epsilon_1\mu^{1/2}h^{1/2}r^{1/2}\gamma^{-1/2} \tag{3.35}$$

this expression is $\asymp Ch^{-2-\delta'}\gamma^3 r^{-1}\ell^4$ and the sum over ℓ-subelements in $B(0,1)$ with ℓ satisfying (3.35) results in $Ch^{-2-\delta'}\gamma^3 r^{-1}$.

Then the sum over all such subelements residing in $\mathcal{Z}_{(r,\gamma)}$ does not exceed

$$Ch^{-2-\delta'}\gamma^3 r^{-1} \times r^2\gamma^{-2} \times r^2\gamma^{-1} = Ch^{-2-\delta'}r^3 \tag{3.36}$$

where the second factor in the left-hand expression are the number of elements which appeared after rescaling (3.26) and the corresponding partition I made in the beginning of subsubsection and the third factor is the number of the "slabs" corresponding to the final elements after the second transformation in the "intermediate" ball $B(0,1)$ obtained after the first transformation.

Then the sum over the whole $\mathcal{Z}^*_{\mathsf{out},I}$ does not exceed $Ch^{-2-\delta'}|\log h|$. So far index p remains fixed; the sum over $p \le C_0/(\mu h)$ does not exceed $C\mu^{-1}h^{-3-\delta'}|\log h| = O(\mu^{-1/2}h^{-3}$. Thus I arrive to

(3.37) The total contribution to the remainder of all $(\mathsf{pair},\mathsf{subelement})$ combinations in zone $\mathcal{Z}^*_{\mathsf{out},I}$ with $\ell \ge \bar{\ell}_1$ does not exceed $C\mu^{-1/2}h^{-3}$.

Therefore only subelements with $\ell\le\bar\ell_1$ remain to be considered.

For such subelements expression (3.34) becomes $C\mu h^{-1-\delta'}\gamma^2\ell^2$; however since without non-degeneracy condition the number of such subelements in $B(0,1)$ could be $\asymp \ell^{-4}$, the sum would become singular. To overcome this obstacle let us redefine the scaling function (3.27) in the manner more suitable for the analysis of elements with $\ell\le\bar\ell_1$:

$$\ell^*=\ell^*(x'',\xi'')=\epsilon\min_{p,n}\Bigl(\gamma^{-2}r^2|\nabla_{x'',\xi''}\mathcal{A}_{pn}|^2+r\gamma^{-1}|\mathcal{A}_{pn}|\Bigr)^{1/2}+\bar\ell_0,\qquad\text{with}\quad \bar\ell_0=\hbar_3^{1/2}h^{-\delta'}=\mu^{-1/2}h^{1/2-\delta''}\gamma^{-3/2}r. \tag{3.38}$$

Note that as the minimum in the right-hand expression is achieved for $(p,n)=(\bar p,\bar n)(x'',\xi'')$, for every index p inequality $\ell\ge\ell_p=\ell^*+\epsilon|p-\bar p|\mu hr$ holds as the $n=n(p,(x'',\xi''))$ is selected to break ellipticity condition; this index $\bar n$ is unique.

Then the contribution of all subelements with $\ell^*\asymp\lambda$ in $B(0,1)$ (obtained after rescaling (3.26)) to the remainder (after summation with respect to n and $p=\bar p$) does not exceed the sum with respect to p of expressions (3.34):

$$C\mu h^{-1-\delta'}\gamma^2\sum_{p\ne\bar p}\bigl(\lambda+\epsilon|p-\bar p|\mu hr\bigr)^{-2}; \tag{3.39}$$

I included in this sum only p with $|p-\bar p|\ge 1$ leaving *special* $((\bar n,\bar p),\mathsf{subelement})$ combinations for a special consideration. Expression (3.39) obviously does not exceed $C\mu^{-1}h^{-3-\delta'}\gamma^2r^{-2}$ and therefore the contribution of all such $(\mathsf{pair},\mathsf{subelement})$ combinations residing in $\mathcal{Z}_{(r,\gamma)}$ does not exceed

$$C\mu^{-1}h^{-3-\delta'}r^{-2}\times r^2\gamma^{-2}\times r^2\gamma^{-1}=C\mu^{-1}h^{-3-\delta'}r^2\gamma^{-1}$$

with the same origin of the second and the third factors in the left-hand expression as in (3.36).

Finally after summation over $\mathcal{Z}^*_{\mathsf{out},l}$ one gets $O(\mu^{-1/2}h^{-3})$. Therefore I arrive to

(3.40) The total contribution to the remainder of all $((p,n),\mathsf{subelement})$ combinations in zone $\mathcal{Z}^*_{\mathsf{out},l}$ with $\ell\le\bar\ell_1$ and $(p,n)\ne(\bar p,\bar n)$ does not exceed $C\mu^{-1/2}h^{-3}$.

3.1.5 So, in what follows I need to consider only *special* $((\bar n,\bar p),\mathsf{subelement})$ combinations with $\ell^*\le\bar\ell_1$. Exactly for these combinations non-degeneracy condition becomes crucial. In the general case however $\mathcal{A}_{pn}$ with $(p,n)=(\bar p,\bar n)$ could be "flat" and then there would be no difference between estimates of the contribution of such subelement to the remainder

and to the main part of the asymptotics which would be $C\hbar_3^{-1}\hbar_4'^{-1}\asymp \mu^2h^{-2}\gamma^5 r^{-2}$. Then the contribution of all such $(\mathsf{pair,subelement})$ combinations residing in $\mathcal{Z}_{(r,\gamma)}$ does not exceed

$$C\mu^2h^{-2}\gamma^5r^{-2}\times r^2\gamma^{-2}\times r^2\gamma^{-1}=C\mu^2h^{-2}\gamma^2r^2 \tag{3.41}$$

with the same origin of the second and the third factors in the left-hand expression as in (3.36).

Finally after summation over $\mathcal{Z}^*_{\mathsf{out},I}$ one gets $O(\mu^{-1/2}h^{-3})$. Therefore I arrive to

(3.42) The total contribution to the remainder of all $((p,n),\mathsf{subelement})$ combinations in zone $\mathcal{Z}^*_{\mathsf{out},I}$ with $\ell\le\bar{\ell}_1$ and $(p,n)=(\bar{p},\bar{n})$ does not exceed $C\mu^2h^{-2}$.

Combining with (3.37),(3.40) I arrive to

Proposition 3.2. *Under condition* (2.1) *the total contribution to the remainder of the zone* $\mathcal{Z}^*_{\mathsf{out},I}$ *does not exceed* $C\mu^2h^{-2}+C\mu^{-1/2}h^{-3}$.

Remark 3.3. All the arguments leading to proposition 3.2 are applicable in $\mathcal{Z}^*_{\mathsf{out},II}$ with the following minor modifications and simplifications:

(i) operators $\mathcal{A}_{pn}$ are $\hbar_3$-PDOs from the very beginning (so there is no need in (3.26) rescaling; there are no "slabs" etc, everything is homogeneous;

(ii) A factor r^2 in estimates translates into γ which makes life easier; in particular there is no need in the special propagation analysis leading to (3.25).

So I arrive to

(i) $O(\mu^{-1/2}h^{-3})$ estimate for the contribution into the remainder of all $((p,n),\mathsf{subelement})$ combinations in zone $\mathcal{Z}^*_{\mathsf{out},II}$ with either $\ell\ge\bar{\ell}_1$ or $\ell\le\bar{\ell}_1$ and $(p,n)\ne(\bar{p},\bar{n})$ and

(ii) $O(\mu^2h^{-2}\gamma^3)$ estimate for the contribution into the remainder of all $((p,n),\mathsf{subelement})$ combinations in zone $\mathcal{Z}^*_{\mathsf{out},II}\cap\{\mathsf{dist}(x,\Sigma)\le\gamma\}$ with $\ell\le\bar{\ell}_1$ and $(p,n)=(\bar{p},\bar{n})$.

In particular:

Proposition 3.4. *Under condition* (2.1) *the total contribution to the remainder of the zone* $\mathcal{Z}^*_{\mathsf{out},II}$ *does not exceed* $C\mu^2h^{-2}+C\mu^{-1/2}h^{-3}$.

3.1.6 Estimate (0.3) $O\bigl(\mu^2h^{-2}+\mu^{-1/2}h^{-3}\bigr)$ is the best possible in the general case (see Appendix A.3) and it coincides with the best possible $O(\mu^{-1/2}h^{-3})$ as $\mu\le h^{-2/5}$; so $\mu\ge h^{-2/5}$ until the end of this subsection.

However, one can do better under some non-degeneracy conditions. Namely, as before one can estimate the contribution of the special pair $((\bar{p},\bar{n}),\text{subelement})$ to the remainder by $C\mu h^{-1-\delta'}\gamma^2\ell^2$ as $\ell\ge\bar{\ell}_0$ (see (3.32)) and by $C\mu^2r^{-2}\gamma^5$ (see (3.33)) where now $\ell=\ell^*$ defined by (3.38). Therefore the contribution to the remainder of all the special subelements residing in the final element $B(0,1)$ (obtained after rescaling/partition (3.26)) does not exceed

$$C\mu h^{-1-\delta'}\gamma^2r^{-2}\int_{\{\bar{\ell}_0\le\ell^*\le\bar{\ell}_1\}}\ell^{*\,-2}\,dx''d\xi''\ +\ C\mu^2h^{-2}r^{-4}\gamma^5\int_{\{\ell^*\le\bar{\ell}_0\}}dx''d\xi'' \tag{3.43}$$

where I remind $\bar{\ell}_0=\mu^{-1/2}h^{1/2-\delta''}\gamma^{-3/2}r$. $\bar{\ell}_1=\mu^{1/2}h^{1/2}r^{1/2}\gamma^{-1/2}$.

Then the total contribution to the remainder of all the special subelements with Ψ-image residing in $\mathcal{Z}_{(r,\gamma)}\subset\mathcal{Z}^*_{\mathsf{out},I}$ to the remainder does not exceed

$$C\mu h^{-1-\delta'}\gamma^2r^{-2}\int_{\{\bar{\ell}_0\le\ell^*\circ\Psi^{-1}\le\bar{\ell}_1\}\cap\mathcal{Z}_{(r,\gamma)}}\ell^{*\,-2}\,|\det D\Psi|^{-1}dx$$
$$C\mu^2h^{-2}r^{-4}\gamma^5\int_{\{\ell^*\circ\Psi^{-1}\le\bar{\ell}_0\}\cap\mathcal{Z}_{(r,\gamma)}}|\det D\Psi|^{-1}dx;$$

Since

$$|\det D\Psi|\asymp\gamma^4r^{-4} \tag{3.44}$$

one can rewrite this expression as

$$C\mu h^{-1-\delta'}\int_{\{\bar{\ell}_0\le\ell^*\circ\Psi^{-1}\le\bar{\ell}_1\}\cap\mathcal{Z}}\gamma^{-2}r^2\ell^{*\,-2}\,dx\ +$$
$$C\mu^2h^{-2}\int_{\{\ell^*\circ\Psi^{-1}\le\bar{\ell}_0\}\cap\mathcal{Z}}\gamma dx \tag{3.45}$$

with $\mathcal{Z}=\mathcal{Z}_{(r,\gamma)}$.

I leave to the reader the similar analysis in $\mathcal{Z}^*_{\mathsf{out},II}$ leading to the same estimate for the contribution of $\bar{\mathcal{Z}}_{(r,\gamma)}\subset\mathcal{Z}^*_{\mathsf{out},II}$ with $r=\gamma^{1/2}$:

Proposition 3.5. *Under condition* (*2.1*)

(i) The total contribution of the special subelements with Ψ*-image residing in* $\mathcal{Z}_{(r,\gamma)}\subset\mathcal{Z}^*_{\mathsf{out},I}$ *does not exceed* (*3.45*) *with* $\mathcal{Z}=\mathcal{Z}_{(r,\gamma)}$*;*

(ii) The total contribution of the special subelements with Ψ-image residing in $\bar{\bar{\mathcal{Z}}}_{(r,\gamma)}\subset\mathcal{Z}^*_{\mathsf{out},II}$ does not exceed (3.45) *with $\mathcal{Z}=\bar{\bar{\mathcal{Z}}}_{(r,\gamma)}$; where $r=\gamma^{1/2}$.*

(iii) The total contribution of all the special subelements (with Ψ-image residing in $\mathcal{Z}^*_{\mathsf{out}}$ does not exceed (3.45) *with $\mathcal{Z}=\mathcal{Z}^*_{\mathsf{out}}$,*

Note that all elements of $D\Psi$ do not exceed $C\gamma r^{-1}$. Combining with (3.45) and (3.16) one can conclude that all elements of $D\Psi^{-1}$ do not exceed $C\gamma^{-1}r$. Therefore if ℓ^* was defined by (3.38) with H_{pn} instead of $\mathcal{A}_{pn}$ then the following inequality would hold

$$\ell^*\circ\Psi^{-1}\ge L\overset{\mathsf{def}}{=}\epsilon|\nabla_\Sigma(V/f_2)|+\min_n|\nabla_\Sigma(V/f_2)-(2n+1)\mu h|. \tag{3.46}$$

Then under condition (0.10) $\ell^*\asymp 1$ even if ℓ^* are defined by $\mathcal{A}_{pn}$ and the first term in (3.44) does not exceed $C\mu h^{-1-\delta'}\gamma^{-1}r^4$ while the second term vanishes and the total contribution of $\mathcal{Z}^*_{\mathsf{out},I}$ to the remainder does not exceed $C\mu^{3/2}h^{-1}$.

Therefore I arrive to

Proposition 3.6. *Under conditions* (2.1) *and* (0.10) *the total contribution of $\mathcal{Z}^*_{\mathsf{out}}$ to the remainder does not exceed $C\mu^{-1/2}h^{-3}$.*

Furthermore, under assumption (3.46) and condition $(0.8)_q$ one can see easily that (3.45) with $\mathcal{Z}=\mathcal{Z}^*_{\mathsf{out}}$ does not exceed $C\mu^{3/2}h^{-1}$ plus

$$C\mu^2h^{-2}\int\gamma\bar{\ell}_0^q\,dx \tag{3.47_q}$$

which is

$$\asymp\iint_\Omega J_q\gamma^{-1}d\gamma\cdot r^{-1}dr,\qquad J_q=h^{-2\delta''}\mu^2h^{-2}(\mu^{-1}hr^2\gamma^{-3})^{q/2}r^2\gamma^2$$

with integral taken over $\Omega=\{\mu^{\delta'-1/2}r^{1/2}\le\gamma\le r^2\le 1\}$.

Obviously $(3.47)_2$ does not exceed J_2 calculated as $\gamma=\mu^{\delta'-1/2}$, $r=1$ which is $O(\mu^{-1/2}h^{-3}$. Further, $(3.47)_1$ does not exceed J_1 calculated as $\gamma=r=1$ which is $C\mu^{3/2}h^{-3/2-\delta''}$ with arbitrarily small $\delta''>0$.

So, one arrives to the following statement (still under (3.46)

Proposition 3.7. *Let condition* (2.1) *be fulfilled. Then*

(i) Under condition $(0.9)_2$ the total contribution to the remainder of the zone $\mathcal{Z}^*_{\mathsf{out}}$ with does not exceed $C\mu^{-1/2}h^{-3}$;

(ii) Under condition $(0.9)_1$ the total contribution to the remainder estimate the zone $\mathcal{Z}^*_{\mathsf{out}}$ does not exceed $C\mu^{3/2}h^{-3/2-\delta}+C\mu^{-1/2}h^{-3}$ with arbitrarily small $\delta''>0$.

Proof. To get rid of assumption (3.26) one needs to take in account perturbation term $\mathcal{B}$ in $\mathcal{A}_{pn}$; then $\ell^*\circ\Psi^{-1}\ge L-|\nabla_\Sigma\mathcal{B}_{pn}\circ\Psi^{-1}|$. Thus it would be enough to replace $\bar{\ell}_0$ by $\bar{\ell}_0+|\nabla_\Sigma\mathcal{B}_{pn}\circ\Psi^{-1}|$ in (3.47).

There are two leading terms in $\mathcal{B}_{pn}$; the first one is $\mu^{-2}r^2\gamma^{-4}((2p+1)\mu h\gamma)^2$; since $p\le C_0/(\mu hr)$ this term does not exceed $C\mu^{-2}\gamma^{-2}$. The second term is $O(\mu^{-1}hr^2\gamma^{-3})$ but looking its origin one can notice that it must contain $|\nabla_\Sigma(V/f_2)|^2$; then

$$|\nabla_\Sigma\mathcal{B}_{pn}\circ\Psi^{-1}|\le L_1\overset{\mathsf{def}}{=}C\mu^{-2}\gamma^{-3}r.$$

Plugging $\bar{\ell}_0=\mu^{-2}\gamma^{-3}r$ into $(3.47)_1$ results in $O(\mu^{1/2}h^{-2})$. □

Remark 3.8. Note that in the extra terms $O(\mu^2h^{-2})$ and $O(\mu^{3/2}h^{-3/2-\delta''})$ appearing in the general case and under condition $(0.8)_q$ the main contributor is far outer zone where $\gamma\approx 1$ and $r\approx 1$ while in the sharp estimate $O(\mu^{-1/2}h^{-3})$ the main contributor is zone where $\gamma\approx\mu^{-1/2}$ and $r\approx 1$;

(ii) Actually one can greatly improve remainder estimate in $(0.8)_1$ case using more refined partition-rescaling technique like in the proof of proposition 4.5. I believe that it can be even brought to $O(\mu^{-1/2}h^{-3})$. However I think that really interesting are only general $(q=0)$ and generic $(q=3)$ cases.

3.2 Estimates in the near outer zone

After deriving remainder estimates in $\mathcal{Z}^*_{\mathsf{out}}$ I need to consider the remaining part of $\mathcal{Z}_{\mathsf{out}}$ (which will be the *near outer zone*

$$\mathcal{Z}'_{\mathsf{out}}=\mathcal{Z}_{\mathsf{out}}\cap\{\mathsf{dist}(x,\Sigma)\le\mu^{-1/2}h^{-\delta}\}\tag{3.48}$$

and the *inner zone*. However I also introduce and start from the *inner core* where I am able to apply pretty non-sophisticated approach and still derive proper estimates, just to keep r more disjoint from 0 to be able to use canonical form (2.38).

3.2.1 In this subsection my goal is to derive remainder estimate $O(\mu^{-1/2}h^{-3}+\mu^2h^{-2})$. Therefore my approach here would be pretty unsophisticated. Note first that

(3.49) The contribution of the domain

$$\bigl\{\mathsf{dist}(x,\Lambda)\le r,\ \mathsf{dist}(x,\Sigma)\le\gamma,\ |Z_1|\le\rho\bigr\}\tag{3.50}$$

to the remainder does not exceed $C\mu h^{-3}\gamma r^2\rho^2$ (while the main part of asymptotics is given by (3.1) with $T_0\ge Ch|\log h|$);

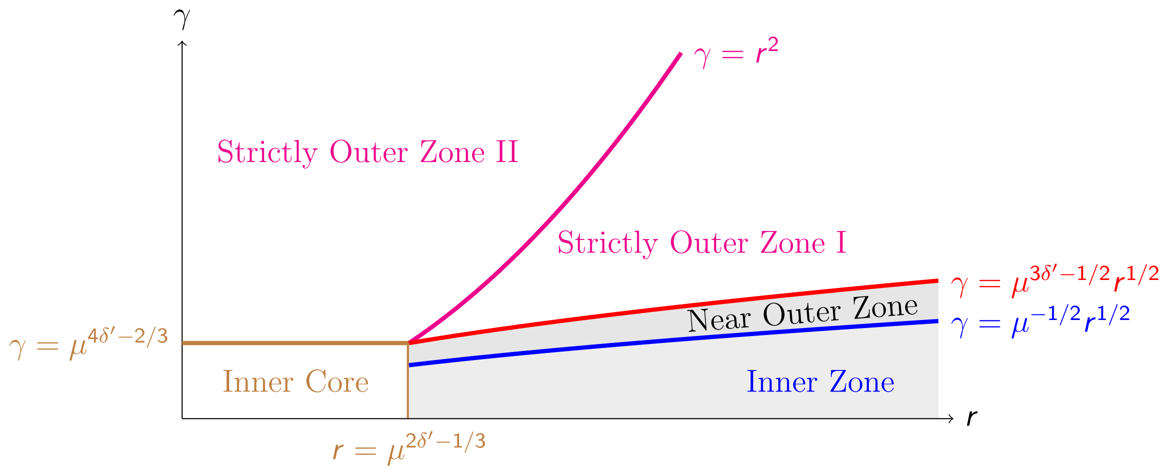

Figure 3: Zones.

therefore considering *inner core* given by (3.50) with $r = \mu^{-1/3+2\delta'}$, $\gamma = \mu^{-2/3+4\delta'}$ one gets $O(\mu^{-1/3+6\delta'} h^{-3})$.

This remainder estimate is very rough and should be improved despite rather unambitious goal of this subsubsection. However let me note first that considering the strip $\{|x_1| \le \mu^{-1/2+3\delta'}\}$ and thus taking $r = 1$, $\rho = 1$ and $\gamma = \mu^{-1/2+3\delta'}$ one would get the remainder estimate $O(\mu^{1/2+3\delta'} h^{-3})$ which is $O(\mu^2 h^{-3})$ as as $\mu \ge h^{-2/3-\delta}$. Further, the contribution of this zone to the asymptotics is $O(h^{-4}\gamma r^2)$ and thus taking $r = 1$, $\rho = 1$ and $\gamma = \mu^{3\delta'-1/2}$ one would get $O(\mu^{-1/2+3\delta'} h^{-4})$ which is $O(\mu^2 h^{-2})$ as $\mu \ge h^{\delta-4/5}$. Therefore in what follows until the end of subsubsection one can assume that

$$h^{-\delta} \le \mu \le h^{\delta-1}. \tag{3.51}$$

Note first that

$$|F_{t\to h^{-1}\tau} \chi_T(t) \Gamma(u Q_y^\dagger)| \le C h^s \tag{3.52}$$

as Q is supported in $\bar{Z}_{(\gamma)} \overset{\mathrm{def}}{=} \{|x_1| \le \gamma\}$ and $T_0 \le T \le T_1$ with $T_0 = Ch|\log h|$ and $T_1 = \epsilon\mu^{-1}$; this statement is empty in the case of the *very strong magnetic field*

$$\epsilon(h|\log h|)^{-1} \le \mu \le Ch^{-1} \tag{3.53}$$

which is not the case now.

However looking at the propagation in direction $\mathbb{K}_1$ as $|Z_1| \asymp \rho$ one can easily notice that the propagation speed is $\asymp \rho$ and the shift for time T is $\asymp \rho T$; this shift is observable as $\rho^2 T \ge Ch|\log h|$. Therefore (3.51) holds as $T_0 \le T \le T_1$ with $T_0 = C\rho^{-2}h|\log h|$ and $T_1 = \epsilon\mu^{-1}\gamma^{-1}$. Under assumption $\gamma \ge C_0\mu^{-1/2}$ this short evolution is contained in $\epsilon_1\gamma$-vicinity of the original point[25].

These two intervals $[Ch|\log h, \epsilon\mu^{-1}]$ and $[C\rho^{-2}h|\log h|, \epsilon\mu^{-1}\gamma^{-1}]$ overlap as

$$\rho \ge (C\mu h|\log h|)^{1/2} \tag{3.54}$$

and therefore I arrive to

(3.55) Under assumptions (3.51) and (3.54) estimate (3.52) holds as $T \in [T_0, T_1]$ with $T_0 = Ch|\log h|,\ T_1 = \epsilon \min\bigl(\mu^{-1}\gamma^{-1}, \mu^{-1/2}\bigr)$.

Therefore the contribution of the domain

$$\bigl\{\mathsf{dist}(x,\Lambda) \le r,\ \mathsf{dist}(x,\Sigma) \le \gamma,\ |Z_1| \ge C(\mu h|\log h|)^{1/2}\bigr\} \tag{3.56}$$

to the remainder does not exceed $C\mu h^{-3}r^2\gamma(\gamma + \mu^{-1/2})$.

Combining with (3.49) I arrive to

Proposition 3.9. *Contribution of domain* (*3.50*) *with* $\rho \asymp 1$ *to the remainder does not exceed*

$$C\mu h^{-3}r^2\gamma(\gamma + \mu^{-1/2}) + C\mu^2 h^{-2}\gamma r^2|\log h| \tag{3.57}$$

while the main part of asymptotics is given by (*0.12*) *with (any)* $T_0 \ge C\mu h|\log h|$.

Remark 3.10. Estimate (3.57) is sufficient for my limited goal here. In particular it covers the inner core giving remainder estimate $O(\mu^{-1/2}h^{-3} + \mu^{2/3+\delta}h^{-2})$ which is not only better than $O(\mu^{-1/2}h^{-3} + \mu^2 h^{-2})$ but than $O(\mu^{-1/2}h^{-3} + \mu^{3/2}h^{-3/2})$.

So, I need to cover the *near outer* and *inner* zones; in both $\gamma \le r^2$ and therefore I can use canonical form (2.38). Even without it, according to section 1 the drift speed is $\asymp \mu^{-1}\rho^2 r\gamma^{-2}$ as $\rho^2 r \ge C_0\gamma$.

Then the shift for time T is observable as $\mu^{-1}\rho^2 r\gamma^{-2}T \times \rho \ge Ch|\log h|$; plugging $T = \mu^{-1}\gamma^{-1}$ one gets $(3.58)_2$ below and therefore *in* (*3.52*) *one can upgrade* $T_1 = \epsilon\mu^{-1}\gamma^{-1}$ *to* $T_1 = \epsilon\mu\gamma^2$ *provided*

$$\rho^2 r \ge c\gamma, \qquad \rho^3 r \ge C\mu^2\gamma^3 h|\log h|. \tag*{$(3.58)_{1,2}$}$$

[25] One can improve this statement and the estimate following from it.

However, the contribution to the remainder of the part of $\mathcal{Z}'_{\mathsf{out}}$ where $T_1 = \epsilon\mu\gamma^2$ is $O(\mu^{-1/2}h^{-3})$. Therefore *one needs to consider only part of $\mathcal{Z}'_{\mathsf{out}}$ where $(3.58)_{1,2}$ is violated*. At this moment one can ignore subzone $\{\rho \le C(\mu h|\log h|)^{1/2}\}$ and therefore one can upgrade $T_1 = \epsilon\mu^{-1}$ to $T_1 = \epsilon\mu^{-1}\gamma^{-1}$ with no penalty.

Moreover, using $O(\mu h^{-3}r^2\gamma^2\rho^2)$ estimate (with $r = 1, \gamma = h^{-1/2+3\delta'}$) one can see easily that the contribution to the remainder of the zone where condition $(3.58)_2$ is violated does not exceed $C\mu^{1/3}h^{-7/3-\delta}$ which $o(\mu^{-1/2}h^{-3} + \mu^2h^{-2})$.

Therefore one needs to consider only zone where condition $(3.58)_1$ is violated; again due to $O(\mu h^{-3}\gamma^2\rho^2r^2)$ estimate its contribution to the remainder does not exceed $C\mu^{-1/2}h^{-3-\delta}$ which is only marginally short of what I want (and only the case $\mu \le h^{-2/5-\delta}$ needs to be considered).

The proper estimate of the zone where condition $(3.58)_1$ is violated can be done easily on the base of proposition 1.6. Really, the drift speed is $\asymp \mu^{-1}(\rho+\rho')\Delta r\gamma^{-2}$ with $\rho' = O(\gamma r^{-1})$, $\Delta = |\rho - \rho'|$ and therefore one can upgrade $T_1 = \epsilon\mu^{-1}\gamma^{-1}$ to $T_1 = \epsilon\mu\gamma^2$ as

$$\mu^{-1}(\rho + \rho')\Delta r\gamma^{-2} \times \mu^{-1}\gamma^{-1} \times \Delta \ge Ch|\log h|$$

with $\rho' = O(\gamma^{1/2}r^{-1/2})$. Therefore one can take

$$\bar{\Delta} = C\mu\gamma^{3/2}r^{-1/2}(h|\log h|)^{1/2}(\rho + \rho')^{-1/2};$$

then as $|\rho - \rho'| \ge \bar{\Delta}$ one can take $T_1 = \epsilon\mu\gamma^2$.

On the other hand, the contribution of the zone $\{(x,\xi) : |\rho - \rho'| \le \bar{\Delta}\} \cap \mathcal{Z}_{(r,\gamma)}$ to the remainder does not exceed

$$C\mu h^{-3}r^2\rho'\gamma r^2\bar{\Delta} = C\mu^2h^{-3}r^2\gamma^{5/2}r^{3/2}(h|\log h|)^{1/2}\rho'^{\,1/2}.$$

Since $\rho' \le c(\gamma/r)^{1/2}$ the latter expression does not exceed $C\mu^2h^{-3}\gamma^{11/4}r^{5/4}(h|\log h|)^{1/2}$. In particular, the contribution to the remainder of the part of $\mathcal{Z}_{(\gamma)}$ where $(3.58)_1$ is violated does not exceed $C\mu^2h^{-5/2}\gamma^{11/4}|\log h|^{1/2}$ and therefore the contribution of the corresponding part of $\mathcal{Z}'_{\mathsf{out}}$ does not exceed $C\mu^{5/8}h^{-5/2-\delta}$.

So I arrive to

Proposition 3.11. *Under condition (2.1) the total contribution to the remainder of zone $\mathcal{Z}'_{\mathsf{out}}$ does not exceed $C(\mu^{-1/2}h^{-3} + \mu^2h^{-2})$.*

3.2.2 Now I need to improve the previous estimate under non-degeneracy condition. However, let us summarize the remainder estimate I actually derived:

(i) The inner core intersected with $\{\rho \ge C(\mu h|\log h|)^{1/2}\}$ contributed $O(\mu h^{-3}\bar{\gamma}_0^2\bar{r}_0^2) = o(\mu^{-1/2}h^{-3})$ where $\bar{\gamma}_0 = \mu^{-2/3+4\delta'}$, $\bar{r}_0 = \mu^{-1/3+2\delta'}$;

(ii) $\mathcal{Z}'_{\mathsf{out}} \cap \{\rho \ge C(\mu h|\log h|)^{1/2}\}$ contributed $O(\mu h^{-3}\bar\gamma_1^2(\mu h\bar\gamma_1)^{2/3}) = O(\mu^{1/3}h^{-7/3-\delta''})$ where $\bar\gamma_1 = \mu^{-1/2+4\delta'}$;

(iii) $\{|x_1| \le \bar\gamma_1\} \cap \{\rho \le C(\mu h|\log h|)^{1/2}\}$ contributed $O(\mu^2 h^{-2}\bar\gamma_1|\log h|) = O(\mu^{1/2+\delta''}h^{-2})$;

each of estimates in (ii),(iii) are less than $O(\mu^{-1/2}h^{-3})$ as $\mu \le h^{-1/2+\delta}$. Therefore in what follows one can assume that

$$h^{-1/2+\delta} \le \mu \le Ch^{-1}. \tag{3.59}$$

The better analysis is based on the canonical form (2.38). However, this requires the inner core to be treated separately:

Proposition 3.12. *Under condition (3.59) the contribution of*

$$\mathcal{Z}^0_{(r)} \overset{\mathrm{def}}{=} \bigl\{\mathsf{dist}(x,\Sigma) \le r^2, \mathsf{dist}(x,\Lambda) \le r\bigr\}$$

with $r \ge \mu^{-1/3+2\delta'}$ does not exceed $C\mu h^{-3}r^6$ while the principal part of asymptotics is given by (0.12) with $T = \bar T \overset{\mathrm{def}}{=} Ch|\log h|$.

Proof. Looking at the canonical form (2.17) and scaling $x'' \mapsto x''r^{-1}$, $x_1 \mapsto x_1 r^{-1}$, $\mu \mapsto \mu_* = \mu r^3$, $h \mapsto h_* = hr^{-1}$ one gets 2-dimensional magnetic Schrödinger operator with parameters μ_*, h_*, which also is 1-dimensional $\hbar_4$-PDO with respect to x_4 with $\hbar_4 = \mu^{-1}hr^{-2}$; for such Schrödinger operators one can prove easily that the main part of asymptotics is given by Weyl formula and is of magnitude $\hbar_4^{-1}h_*^{-2} = \mu h^{-3}r^4$ while the remainder estimate gains the factor $\mu_* h_* = \mu h r^2$ and thus is of magnitude $\mu^2 h^{-2} r^6$. Really, if $\mu_* = 1$ it would follow from the standard results (and no non-degeneracy condition is needed); as $\mu_* \ge 1$ one needs just apply extra rescaling $x' \mapsto x'\mu_*$ which is also rather standard. $\square$

Corollary 3.13. *Therefore even without nondegeneracy conditions contribution of the (extended) inner core $\mathcal{Z}^0_{(r)}$ with $r = \mu^{-1/4}$ to the remainder does not exceed $C\mu^{-1/2}h^{-3}$.*

Thus in what follows I can apply canonical form (2.38). Let us fix some index n and some final $B(0,1)$ element where in contrary to subsection 3.1 final means only after reduction to (2.38). This element is either *regular*, when on the original $B(\bar x', \epsilon r)$ element

$$|Vf_2^{-1} - (2n+1)\mu h| + |\nabla_{x'}\bigl(Vf_2^{-1} - (2n+1)\mu h\bigr)|^2 \asymp L^2 \ge r^2 \tag{3.60}$$

or *singular* when on the original $B(\bar x', \epsilon r)$ element

$$|Vf_2^{-1} - (2n+1)\mu h| + |\nabla_{x'}\bigl(Vf_2^{-1} - (2n+1)\mu h\bigr)|^2 \le r^2. \tag{3.61}$$

Since analysis of the regular elements is similar to the analysis of the singular elements but simpler, I consider only singular elements, leaving regular elements to the reader.

Let us introduce the scaling function with respect to (x_3, x_4, ξ_4)

$$\ell = \ell_n = \epsilon r^{-1}|\nabla_{x_3,x_4,\xi_4} V f_2^{-1}| + \bar{\ell} \tag{3.62}$$

where $\bar{\ell} \geq C\gamma r^{-1}|\log h|$ will be chosen later. With respect to $|Z_1^\#|$ I use scaling function $\rho = \epsilon\bigl(|Z_1| + r^{1/2}\ell\bigr)$. Then ellipticity of $\mathcal{A}_n$ is not broken unless $\bigl(Vf_2^{-1} - (2n+1)\mu h\bigr) \asymp \rho^2$ or $\rho \leq cr^{1/2}\ell$.

Then it follows from the standard theory that for u_n holds as Q is supported in the corresponding ℓ-partition element, $T \in [T_0, T_1]$, $\tau \leq Cr\ell^2$ with

$$T_0 = Ch\bigl(\rho^2 + r\ell^2\bigr)^{-1}|\log h|, \tag{3.63}$$

$$T_1 = \epsilon\mu^{-1}\gamma^{-1}, \tag{3.64}$$

$$\rho + r^{1/2}\ell \geq C\bar{\rho}_0 \overset{\mathrm{def}}{=} C(\mu h\gamma|\log h|)^{1/2} \tag{3.65}$$

and moreover, under these conditions

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma(u_n Q y^\dagger)| \leq C\mu h^{-2} r^3\ell^3\gamma \tag{3.66}$$

Therefore

(3.67) While the contribution of this $(\ell;\rho)$-subelement into asymptotics does not exceed $C\mu h^{-3}r^3\bigl(\rho^2 + r\ell^2\bigr)\ell^3\gamma$ its contribution to the remainder so far does not exceed

$$C\mu h^{-3}r^3\bigl(\rho^2 + r\ell^2\bigr)\ell^3\gamma \times T_0 T_1^{-1} = C\mu^2 h^{-2} r^3\ell^3\gamma^2.$$

Note that for each $(\ell;\rho)$-partition subelement in $(x_3, x_4, \xi_4; \xi_1, \xi_3)$ the number of indices n satisfying (3.62) and also violating ellipticity does not exceed

$$M \overset{\mathrm{def}}{=} \Bigl(C_0(\mu h)^{-1}(\rho^2 + r\ell^2) + 1\Bigr) \tag{3.68}$$

Also note that under condition $(0.8)_q$ the total number of ℓ-subelements is $O(\ell^{q-3})$. Then the total contribution of all ℓ-elements to the asymptotics (as magnitudes of r, γ are given) does not exceed

$$C\mu h^{-3}\gamma r^{(2-q)_+}(\rho^2 + r\ell^2) \times \Bigl(C_0(\mu h)^{-1}(\rho^2 + r\ell^2) + 1\Bigr) \times \ell^q \tag{3.69}$$

which one can sum nicely to its value at maximal $\ell=1$, $\rho=1$ and then to maximal $r=1$, $\gamma=\mu^{\delta-1/2}$ resulting $Ch^{-4}\mu^{\delta-1/2}$.

On the other hand, the total contribution of ℓ-subelements to the remainder so far does not exceed

$$C\mu^2 h^{-2}\gamma^2 r^{(2-q)_+}\times \Bigl(C_0(\mu h)^{-1}(\rho+r\ell)^2+1\Bigr)\times \ell^q \tag{3.70}$$

which one can also sum nicely according to the same principles but the sum is $C\mu h^{-3}\gamma^2$ not estimated that well and ℓ, r close to 1 are problematic. So I want to increase T_1.

I remind that $W_n=\omega\bigl(Vf_2^{-1}-(2n+1)\mu h\bigr)$ with $\omega=(\sigma\sigma_1)^{-1}f_2$ and here (see proposition 2.8) $\omega\sim|\{Z_1,f_1\}|^{-2/3}$ rescaled; therefore for "small" ρ calculation of ∇W_n and $\omega\nabla\bigl(Vf_2^{-1}-(2n+1)\mu h\bigr)$ are equivalent modulo $\omega\rho^2 r^{-1}$. Note that

(3.71) As $\bigl(Vf_2^{-1}-(2n+1)\mu h\bigr)\asymp\rho^2$ and $\ell\ge C\rho^2 r^{-1}$ the propagation speed with respect to x_4 does not exceed $C\mu^{-1}r^{-1}\ell$ and is of this magnitude as $|\partial_{\xi_4}W_n|\asymp\ell$; then the shift with respect to x_4 is observable as

$$\mu^{-1}r^{-1}\ell T\times\ell\ge C\mu^{-1}r^{-2}h|\log h|; \tag{3.72}$$

(3.73) As $\bigl(Vf_2^{-1}-(2n+1)\mu h\bigr)\asymp\rho^2$ and $\ell\ge C\rho^2 r^{-1}$ the propagation speed with respect to $\mu^{-1}hD''$ does not exceed $C\mu^{-1}r^{-1}\ell$ and is of this magnitude as $|\partial_{x''}W_n|\asymp\ell$; then the shift with respect to $\mu^{-1}hD''$ is observable under the same condition;

(3.74) Finally, in the short-term propagation (as $T\le\epsilon\mu^{-1}\gamma^{-1}$) the propagation speed with respect to (x_1,x_3) is of magnitude $|Z_1|\asymp\rho$ and the shift is observable as $\rho T\times\rho\ge Ch|\log h|$.

On the other hand,

(3.75) Let $\rho^2\ge C\gamma$; then the drift speed (in the rescaled coordinates) with respect to x'' is of magnitude $C\mu^{-1}\gamma^{-2}\rho^2$ and the shift for time T is observable as

$$\mu^{-1}\gamma^{-2}\rho^2 T\times\rho\ge Chr^{-1}|\log h|. \tag{3.76}$$

Therefore one can increase $T_1 = \epsilon\mu^{-1}\gamma^{-1}$ provided one of the conditions (3.72),(3.76) holds with $T = T_1$. Plugging $T_1 = \epsilon\mu^{-1}\gamma^{-1}$ into (3.72), (3.76) I get

$$\rho \ge \bar{\rho} \stackrel{\mathsf{def}}{=} C\bigl(\mu^2 h\gamma^3 r^{-1}|\log h|\bigr)^{1/3} + C\gamma^{1/2}, \tag{3.77}$$

$$\ell \ge \bar{\ell} \stackrel{\mathsf{def}}{=} C\bigl(\mu h\gamma r^{-1}|\log h|\bigr)^{1/2} + C\rho^2 r^{-1} \tag{3.78}$$

respectively.

Now consider increased T_1. I claim that

(3.79) Under condition (3.78) one can take $T = \epsilon\mu\gamma^2\ell$.

Really, then magnitude of $|\nabla_{x_1,x_4,\xi_4} W_n|$ changes by no more than $\epsilon\ell$ as long as $\rho \le \epsilon_1$; however, as $\rho \ge \epsilon$ condition (3.77) is fulfilled and one can take $T_2 = \epsilon\mu\gamma^2$;

Therefore the contribution to the remainder of the zone with the given magnitudes of ρ, ℓ, γ, r satisfying (3.78) does not exceed expression (3.69) multiplied by $T_0 T_2^{-1}$; the resulting expression does not exceed

$$\begin{gathered} C\mu h^{-3}\gamma r^{(2-q)_+}(\rho^2 + r\ell^2) \times \Bigl(C_0(\mu h)^{-1}(\rho + r\ell)^2 + 1\Bigr) \times \ell^q \times \\ h(\rho^2 + r\ell^2)^{-1} \times \mu^{-1}\gamma^{-2}\ell^{-1} = \\ C\mu^{-1}h^{-3}\gamma^{-1}r^{(2-q)_+}\Bigl((\rho + r\ell)^2 + \mu h\Bigr)\ell^{q-1}; \end{gathered} \tag{3.80}$$

then summation with respect ℓ, ρ results in $C\mu^{-1}h^{-3}\gamma^{-1}r$ as $q > 1$ (then one should take q slightly larger than 1). As $q = 1$ one gets $C\mu^{-1/2}h^{-3}(1 + \mu h|\log h|)$ as provided summation is taken over $\rho \le c(\ell + |\log h|^{-1})$; but the latter case is covered by arguments below.

(3.81) Under conditions (3.77) and $\rho \ge r^{1/2}\ell$ one can take $T_2 = \epsilon\mu\gamma^2\rho r^{-1/2}$.

Really, in this case due to bounds for propagation and drift speeds this guarantees that both $|\nabla_{x_1,x_4,\xi_4} W_n|$ and $W_n \asymp |Z_1|^2$ change by no more than $r^{-1/2}\rho$, $\epsilon_1\rho^2$ respectively during the evolution.

Now instead of (3.80) one gets

$$\begin{gathered} C\mu h^{-3}\gamma r^{(2-q)_+}(\rho^2 + r\ell^2) \times \Bigl(C_0(\mu h)^{-1}(\rho + r\ell)^2 + 1\Bigr) \times \ell^q \times \\ h(\rho^2 + r\ell^2)^{-1} \times \mu^{-1}\gamma^{-2}\rho^{-1} \le \\ C\mu^{-1}h^{-3}\gamma^{-1}r^{(2-q)_+}\Bigl((\rho + r\ell)^2 + \mu h\Bigr)\ell^q\rho^{-1} \end{gathered} \tag{3.82}$$

and summation with respect to ρ, ℓ, γ, r results in $C\mu^{-1/2}h^{-3}$. Therefore

(3.83) The total contribution to the remainder of elements in $\mathcal{Z}''_o ut$ satisfying (3.65) and one of conditions (3.77), (3.78) does not exceed $C\mu^{-1/2}h^{-3}$.

3.2.3 So, I am left with two types of elements, which are not covered by the above arguments:

(3.84) With $\bar\rho_0\le\rho\le\bar\rho=C(\mu^2h\gamma^3r^{-1}|\log h|)^{1/3}+C\gamma^{1/2}$ and $\ell\le\bar\ell\asymp r^{-1}\bar\rho_0+\rho^2r^{-1}$;

(3.85) With $\rho\le\bar\rho_0=C(\mu h\gamma)^{1/2}$ and $\ell\le\bar\ell\asymp r^{-1/2}\bar\rho_0$.

For (3.84)-type elements $r\ell^2\le\rho^2$ and expression (3.69) becomes

$$C\mu h^{-3}\gamma r^{(2-q)_+}\rho^2\times\Bigl((\mu h)^{-1}\rho^2+1\Bigr)\times\bar\ell^q;$$

multiplying it by $T_0T_1^{-1}=Ch\rho^{-2}|\log h|\cdot\mu\gamma$ one gets

$$C\mu^2h^{-2}\gamma^2r^{(2-q)_+}\times\Bigl((\mu h)^{-1}\rho^2+1\Bigr)\times\bar\ell^q|\log h|$$

which after summation over ρ becomes

$$C\mu^2h^{-2}\gamma^2r^{(2-q)_+}\times\Bigl((\mu h)^{-1}\bar\rho^2+1\Bigr)\times\bar\ell^q|\log h|. \tag{3.86}$$

Meanwhile for (3.85)-type elements expression (3.69) becomes

$$C\mu h^{-3}\gamma r^{(2-q)_+}\bar\rho_0^2\bar\ell^q. \tag{3.87}$$

Let us plug $\bar\rho$, $\bar\rho_0$, $\bar\ell$ and γ; one can take

$$\gamma=\mu^{-1/2}r^{1/2},\ \ \bar\rho_0=(\mu hr^{1/2})^{1/2},\bar\rho=(\mu^{1/2}hr^{1/2})^{1/3}+\mu^{-1/4}r^{1/4}\quad\text{and}\\ \bar\ell=(\mu^{1/2}hr^{-1/2})^{1/2}+(\mu^{1/2}h)^{2/3}r^{-2/3}+\mu^{-1/2}r^{-1/2},$$

and compensate this by multiplication the result by $h^{-\delta''}$ with an arbitrarily small exponent $\delta''>0$.

Take first $\bar\ell=\bar\ell_1\stackrel{\mathsf{def}}{=}(\mu^{1/2}hr^{-1/2})^{1/2}$; note that then in (3.86), (3.87) factor $r^{-q/4}$ coming from $\bar\ell^q$ is more than compensated by the factor r coming from γ^2 or $\gamma\bar\rho_0^2$ respectively. Therefore summation with respect to γ,r results in the same expressions calculated as $r=1$, $\gamma=\mu^{-1/2}$, $\bar\ell=\bar\rho_0=(\mu^{1/2}h)^{1/2}$ and $\bar\rho=(\mu^{1/2}h)^{1/3}+\mu^{-1/4}$:

$$Ch^{-3}\Bigl((\mu^{1/2}h)^{2/3}+\mu^{-1}+\mu h\Bigr)(\mu^{1/2}h)^{q/2}\times h^{-\delta''}, \tag{3.88}$$

$$C\mu^{1/2}h^{-3}(\mu^{1/2}h)^{(2+q)/2}\times h^{-\delta''}. \tag{3.89}$$

One can see easily that both these expressions do not exceed

(i) $C\bigl(\mu^{-1/2}h^{-3}+\mu^{3/2}h^{-3/2}\bigr)$ as $q=1$;

(ii) $C\bigl(\mu^{-1/2}h^{-3}+\mu^{1/2+\delta''}h^{-2}\bigr)$ as $q=2$;

(iii) $C\mu^{-1/2}h^{-3}$ as $q=3$.

However one should consider (3.86) with $\bar\ell=\bar\ell_2\overset{\mathsf{def}}{=}(\mu^{1/2}h)^{2/3}r^{-2/3}$ and $\bar\ell=\bar\ell_3\overset{\mathsf{def}}{=}\mu^{-1/2}r^{-1/2}$; as $q=1$ one can take again $r=1$ obviously; as $q=2$ it is not so but one can take then by $q=9/5$ arriving to $O(\mu^{-1/2}h^{-3})$.

Then combining with (3.83) I arrive to

Proposition 3.14. *The total contribution to the remainder of the zone $\mathcal{Z}'_{\mathsf{out}}$ does not exceed $C\mu^{-1/2}h^{-3}+C\mu^{3/2}h^{-3/2}$ under condition $(0.8)_1$, $C\mu^{-1/2}h^{-3}+C\mu^{3/2}h^{-1-\delta}$ under condition $(0.8)_2$ and $C\mu^{-1/2}h^{-3}$ under condition $(0.8)_3$.*

Remark 3.15. The main part of this asymptotics is given by expression (0.12) with $T=\bar T_0$.

3.3 Estimates in the inner zone

Now I want to consider the *inner zone*

$$\mathcal{Z}_{\mathsf{inn}}\overset{\mathsf{def}}{=}\bigl\{\mathsf{dist}(x,\Sigma)\le\bar\gamma_0=c\mu^{-1/2}\cdot\mathsf{dist}(x,\Lambda)^{1/2}\bigr\},\tag{3.90}$$

in term consisting off the *inner bulk zone*

$$\mathcal{Z}_{\mathsf{inn},II}\overset{\mathsf{def}}{=}\mathcal{Z}_{\mathsf{inn}}\cap\bigl\{|Z_1|\cdot\mathsf{dist}(x,\Lambda)\le\epsilon\mu\cdot\mathsf{dist}(x,\Sigma)^2\bigr\}\tag{3.91}$$

which mimics the outer (bulk) zone $\mathcal{Z}_{\mathsf{out}}$ and the *true inner zone*

$$\mathcal{Z}_{\mathsf{inn},I}\overset{\mathsf{def}}{=}\mathcal{Z}_{\mathsf{inn}}\cap\bigl\{|Z_1|\cdot\mathsf{dist}(x,\Lambda)\ge\epsilon\mu\cdot\mathsf{dist}(x,\Sigma)^2\bigr\}\tag{3.92}$$

which mimics the inner zone for 2-dimensional Schrödinger operator. One needs to remember that trajectories can leave one of them and enter another one after a while.

Note first that the inner core is already covered by proposition 3.12, so in the definition of the inner zone I can include condition $\mathsf{dist}(x,\Lambda)\ge\mu^{-1/4}$ and therefore use canonical form (3.28) there. Actually using the same method as in the proof of proposition 3.12, one can provide estimate $O(\mu h^{-3}r^2\gamma^2)=O(h^{-3}r^3)$ thus covering even $\mathcal{Z}_{\mathsf{inn}}\cap\{\mathsf{dist}(x,\Lambda)\le\mu^{-1/6}\}$ but I don't need this.

3.3.1 I start from the inner bulk zone $\mathcal{Z}_{\mathsf{inn},II}$. Let $|Z_1| \asymp \rho$, $\rho r \le \epsilon\mu\gamma^2$ and condition (3.77) be fulfilled. Then exactly as in the outer zone the drift speed is $\asymp \mu^{-1}\rho^2 r\gamma^{-2} \le c\rho$ and evolution speed with respect to hD_4 does not exceed c; then for time $T_1 = \epsilon\rho r$ dynamics remains in $\mathcal{Z}_{\mathsf{inn},II}$. Therefore in the domain where condition (3.77) is fulfilled and $\rho \ge C(\mu h|\log h|)^{1/2}$ one can take $T_1 = \epsilon\rho r$ and $T_0 = Ch|\log h|$; however in the estimates this logarithmic factor is not needed.

Therefore the contribution to the remainder of this domain intersected with $\mathcal{Z}_{(r,\gamma,\rho)}$ does not exceed $Ch^{-3}\gamma T_1^{-1} r^2\rho^2 = C\mu^{-1/2}h^{-3}r^2\rho$ and the summation with respect to (r,γ,ρ) results in $O(\mu^{-1/2}h^{-3})$.

On the other hand, consider the domain where condition (3.77) is violated, but still $\rho \ge \bar\rho_1$. The contribution to the remainder of of this domain intersected with $\mathcal{Z}_{(r,\gamma,\rho)}$ does not exceed $C\mu h^{-3}\gamma^2 r^2\rho^2$ where two factors γ come as the width of the strip and the part of $T_1^{-1} \asymp \mu\gamma$. The summation with respect to (r,γ,ρ): $r\rho^2 \le c\gamma$ results in $C\mu^{-1/2}h^{-3}$.

Further, consider domain where condition (3.78) is violated but still $\rho \ge \bar\rho_1$. In the similar manner one can estimate its contribution to the remainder by $C\mu h^{-3}\gamma_0^2(\mu^{1/2}h|\log h|)^{2/3}$ which is $o(\mu^2h^{-2} + \mu^{-1/2}h^{-3})$ for sure and is $O(\mu^{-1/2}h^{-3})$ as $\mu \le C(h|\log h|)^{-4/5}$.

Finally, contribution to the remainder of the domain where $\rho \le \bar\rho_1$ does not exceed $C\mu\gamma\bar\rho_1^2 = C\mu^{3/2}h^{-2}|\log h|$.

Thus I arrive to

Proposition 3.16. *Contribution of $\mathcal{Z}_{\mathsf{inn},II}$ to the remainder is $O(\mu^2h^{-2} + \mu^{-1/2}h^{-3})$ for sure. Furthermore, this contribution is is $O(\mu^{-1/2}h^{-3})$ as $\mu \le C(h|\log h|)^{-1/2}$. Again the main part of this asymptotics is given by expression (0.12).*

3.3.2 Again, the estimate achieved in proposition 3.16 is good in the general case but should be improved under non-degeneracy condition as

$$c(h|\log h|)^{-1/2} \le \mu \le Ch^{-1}. \tag{3.93}$$

Again one should consider regular final elements and singular ones and I consider only latter. Further introducing the scaling function ℓ_n by (3.62) I can define T_0 and T_1 by (3.63),(3.64) as (3.65) holds; furthermore estimate (3.66) holds.

Further, if condition (3.78) is fulfilled, I can increase T_1 to $T_2 = \epsilon\ell$ since on this time interval both $|\nabla_{x_1,x_4,\xi_4}|$ change by no more than $\epsilon_1 \min(\ell, r)$.

Then repeating the arguments of subsubsections 3.2.2 I find that the contribution of this part of $\mathcal{Z}'_{\mathsf{inn},II}$ to the remainder does not exceed expression similar to (3.80) but with

$\mu^{-1}\gamma^{-2}$ replaced by 1:

$$C\mu h^{-3}\gamma r^{(2-q)_+}(\rho^2+r\ell^2)\times\Bigl(C_0(\mu h)^{-1}(\rho+r\ell)^2+1\Bigr)\times\ell^q\times h(\rho^2+r\ell^2)^{-1}\times\ell^{-1}=\\ Ch^{-3}\gamma r^{(2-q)_+}\Bigl((\rho+r\ell)^2+\mu h\Bigr)\ell^{q-1};\tag{3.94}$$

Then the sum with respect to (ℓ,ρ,r)-partition with $\rho\ge\ell$ does not exceed $Ch^{-3}\gamma$ as $q\ge 2$ and $Ch^{-3}\gamma|\log h|$ as $q=1$ and one can get rid off logarithmic factor exactly as in the analysis in $\mathcal{Z}'\mathsf{out},II$.

Furthermore, condition (3.77) is fulfilled instead of (3.78) and $\rho\ge R^{1/2}\ell$, I can increase T_1 to $T_2=\epsilon\rho$ and the contribution of this part of $\mathcal{Z}'_{\mathsf{inn},II}$ to the remainder does not exceed expression similar to (3.82) but with $\mu^{-1}\gamma^{-2}$ replaced by 1:

$$C\mu h^{-3}\gamma r^{(2-q)_+}(\rho^2+r\ell^2)\times\Bigl(C_0(\mu h)^{-1}(\rho+r\ell)^2+1\Bigr)\times\ell^q\times h(\rho^2+r\ell^2)^{-1}\times\rho^{-1}=\\ Ch^{-3}\gamma r^{(2-q)_+}\Bigl(\rho^2+\mu h\Bigr)\ell^q\rho^{-1};\tag{3.95}$$

then summation with respect to (ℓ,ρ,r)-partition results in $Ch^{-3}\gamma+C\mu h^{-2}\gamma|\log h|$. Therefore the sum with respect to γ-partition with $\rho\ge\ell$ does not exceed $C\mu^{-1/2}h^{-3}$ as $q\ge 2$ and $\mu^{-1/2}h^{-3}+C\mu^{1/2}h^{-2}|\log h|$ as $q=1$.

On the other hand, due to the same arguments as in subsubsection 3.2.3 the contribution to the remainder of the domain where both conditions (3.77),(3.78) are violated but l (3.65) still holds does not exceed (3.86). Then the summation with respect to (ℓ,ρ,r,γ) partition results again in $C\mu h^{-2}(\mu^{1/2}h|\log h|)^{q/2}+O(\mu^{-1/2}h^{-3})$.

Finally, contribution of the domain where (3.65) fails is tackled in the same way as in subsubsection in subsubsection 3.2.3 and estimated by (3.87).

So, I arrive to the following copy-cat of proposition 3.14 and remark 3.15:

Proposition 3.17. *The total contribution to the remainder of the zone* $\mathcal{Z}_{\mathsf{inn},II}$ *does not exceed* $C\mu^{-1/2}h^{-3}+C\mu^{3/2}h^{-3/2}$ *under condition* $(0.8)_1$, $C\mu^{-1/2}h^{-3}+C\mu^{3/2}h^{-1}|\log h|$ *under condition* $(0.8)_2$ *and* $C\mu^{-1/2}h^{-3}$ *under condition* $(0.8)_3$.

Again the main part of this asymptotics is given by expression (0.12).

3.3.3 Now let us consider the *true inner zone* $\mathcal{Z}_{\mathsf{inn},I}$ defined by (3.94). The crucial difference between this zone and $\mathcal{Z}_{\mathsf{out}}\cap\mathcal{Z}_{\mathsf{inn},II}$ is that in the classical evolution $\mathsf{dist}(x,\Sigma)$ is not necessarily preserved[26)]

[26)] As $|Z_1|\cdot\mathsf{dist}(x,\Lambda)\ge C\mu\cdot\mathsf{dist}(x,\Sigma)^2$ it is not preserved for sure; in the rest of this zone variation of $\mathsf{dist}(x,\Sigma)$ is of the magnitude of $\mathsf{dist}(x,\Sigma)$.

I remind that this subzone (as the whole inner zone) should be studied as $r=\mathsf{dist}(x,\Lambda)\ge c\mu^{-1/4}$. Also I remind that the contribution of subzone $\{|Z_1|\le C(\mu h\log h)^{1/2}\}$ to the remainder does not exceed $C\mu h^{-3}\gamma\times\mu h|\log h| = C\mu^{3/2}h^{-2}|\log h|)$ which is $O(\mu^2h^{-2})$ for sure and $O(\mu^{-1/2}h^{-3})$ as $\mu\le c(h|\log h|)^{-1/2}$.

Let us introduce $\gamma=\mu^{-1/2}(r\rho)^{1/2}$; then in the classical evolution γ is the magnitude of $\max\mathsf{dist}(x,\Sigma)$ along the cyclotron movement. Note that the contribution of subzone $\{(\mu h|\log h|)^{1/2}\le|Z_1|\le\rho\}$ to the remainder does not exceed $C\mu h^{-3}\gamma^2\rho^2 = Ch^{-3}\rho^3$ which sums to $O(\mu^{-1/2}h^{-3})$ in subzone $\{\rho\le c\mu^{-1/6}|\log h|^{-1/3}\}$.

Consider first the general case of $q=0$ when our goal is $O(\mu^{-1/2}h^{-3}+\mu^2h^{-2})$ estimate. Then the total contribution of $\mathcal{Z}_{\mathsf{inn},l}\cap\{|Z_1|\ge C(\mu h|\log h|)^{1/2}\}$ to the remainder is $O(h^{-3})$. Therefore in this case one needs to analyze only $\mu\le h^{-1/2}$ and only subzone $\{\rho\ge c(\mu^{-1/6}+\mu^{2/3}h^{1/3})\}$.

I am going to prove that the contribution of $\mathcal{Z}_{\mathsf{inn},l}\cap\{\rho\ge C(\mu h|\log h|)^{1/2}\}$ to the remainder does not exceed $C\mu^{-1/2}h^{-3}+C(\mu^{1/2}h|\log h|)^{1/2}h^{-3}$ (see proposition 3.19) but I prefer a bit less direct way to do this; I am concentrating on this zone due to the previous remarks and because I do not have non-degeneracy condition at the moment. Thus one can take $T_0=Ch|\log h|$ and $T_1=\epsilon\mu^{-1/2}\gamma^{-1}=\epsilon\mu^{-1}(\rho r)^{-1}$. Let us try to increase it.

According to the series of propositions of subsubsection 1.4.3 the drift speed in the inner zone is $\kappa r\rho(\xi_3-k^*\rho)+O(\mu^{-1/2})$. Then the shift during one cyclotron movement is $\asymp r\rho(\xi_3-k^*\rho)T_1$ provided

$$r\rho|\xi_3-k^*\rho|\ge C\mu^{-1/2} \tag{3.96}$$

and it is observable as

$$r\rho|\xi_3-k^*\rho|\times\mu^{-1/2}\times|\xi_3-k^*\rho|\ge Ch|\log h|$$

or equivalently

$$|\xi_3-k^*\rho|\ge C(r\rho)^{-1/2}(\mu h\gamma|\log h|)^{1/2}=C(r\rho)^{-1/4}(\mu^{1/2}h|\log h|)^{1/2}. \tag{3.97}$$

Both conditions (3.96), (3.97) are fulfilled as

$$|\xi_3-k^*\rho|\ge\Delta\overset{\mathsf{def}}{=}c(r\rho)^{-1/4}(\mu^{1/2}h|\log h|)^{1/2}+c\mu^{-1/2}(r\rho)^{-1/2}. \tag{3.98}$$

Then as (3.98) is fulfilled[27] one can upgrade T_1 to $T_2=\epsilon r|\xi_3-k^*\rho|$ because then not only magnitude of ρ but also of $|\xi_3-k^*\rho|$ will be preserved during the evolution. However

[27] This makes sense as $\Delta\le\rho$: i.e.

$$\rho\ge\bar\rho_2\overset{\mathsf{def}}{=}r^{-1/5}(\mu^{1/2}h|\log h|)^{2/5}+\mu^{-1/3}r^{-1/3}; \tag{3.99}$$

otherwise one needs to take $\Delta=c\rho$, thus redefining $\Delta\overset{\mathsf{redef}}{=}\min(\Delta,\bar\rho_2)$.

taking the evolution in the right time direction it could be increased to $T_2 = \epsilon r|\xi_3 - k^*\rho|^{1-\delta'}$ exactly as in [Ivr6]. The standard calculations imply

(3.100) The contribution of subzone of $\mathcal{Z}_{\mathsf{inn},I}$ where condition (3.98) is fulfilled intersected with $\{\rho \ge C(\mu h|\log h|)^{1/2}\}$ to the remainder does not exceed $C\mu^{1/2}h^{-3}$.

So, one needs to study *near periodic zone*

$$\mathcal{Z}_{\mathsf{per}} = \bigl\{|\xi_3 - k^*\rho| \le \Delta\bigr\}, \tag{3.101}$$

intersected with $\{|Z_1| \asymp \rho\} \cap \{\mathsf{dist}(x,\Lambda) \asymp r\}$ as

$$\rho \ge C(\mu h|\log h|)^{1/2} + \bar{\rho}_2.$$

Note however that as $r \le \epsilon$ then $|\nabla\omega| \asymp r^{-1}\omega$ and $|\operatorname{Hess}\omega| \asymp r^{-2}\omega$ and therefore in the rescaled coordinates $|\nabla_{x_3,x_4,\xi_4} W_n| \asymp \rho^2$ as $r \le \epsilon$ and $\rho^2 \ge Cr$ and $|\operatorname{Hess}_{x_3,x_4,\xi_4} W_n| \asymp \rho^2$ as $r \le \epsilon$ and $\rho \ge Cr$. In the first case periodicity is broken for sure and in the second case periodicity is not broken only as $|\nabla W_n| \le C(\mu\gamma h)^{1/2}$ which in intersection of $\mathcal{Z}_{\mathsf{per}}$ has measure not exceeding $C\mu^{1/2}h\gamma$ instead of $C(\mu^{1/2}h)^{1/2}\gamma$ and since one can take $T_1 = \epsilon\mu^{-1}\gamma^{-1}$ here for sure the contribution of the reduced periodic zone to the remainder estimate does not exceed $Ch^{-4} \times (\mu^{1/2}h|\log h|)\gamma \times \mu h\gamma \le C\mu^{1/2}h^{-2}|\log h|$ thus implying

Proposition 3.18. *The contribution of $\mathcal{Z}_{\mathsf{inn},I} \cap \{|Z_1| \ge C(\mu h|\log h|)^{1/2} + Cr\}$ to the remainder does not exceed $C\mu^{-1/2}h^{-3}$.*

Again the main part of this asymptotics is given by expression (0.12).

So, only $\rho \le Cr$ and thus $\gamma \le C\mu^{-1/2}r$ should be analyzed.

Proposition 3.19. *The contribution of $\mathcal{Z}_{\mathsf{inn},I} \cap \{|Z_1| \ge C(\mu h|\log h|)^{1/2}\}$ to the remainder does not exceed $C\mu^{-1/2}h^{-3} + C(\mu^{1/2}h|\log h|)^{1/2}h^{-3}$.*

Again the main part of this asymptotics is given by expression (0.12).

Proof. Now I am concentrating only on even more reduced zone. In particular (3.99) implies $r \ge (\mu^{1/2}h|\log h|)^{1/3} + \mu^{-1/4}$.

One can take $T_1 = \epsilon\mu^{-1/2}(r\rho)^{-1/2}$ as condition (3.98) fails. Then the contribution of $\mathcal{Z}_{\mathsf{per}}$ to the remainder does not exceed

$$\begin{aligned} &C\mu h^{-3}\gamma^2 r^2\rho\Delta = Ch^{-3}r^3\rho^2\Delta = \\ &\qquad Ch^{-3}r^3\rho^2\Bigl(c(r\rho)^{-1/4}(\mu^{1/2}h|\log h|)^{1/2} + c\mu^{-1/2}(r\rho)^{-1/2}\Bigr). \end{aligned} \tag{3.102}$$

Obviously when taking sum with respect to (r,ρ) partition the latter expression sums to itself with $\bar\Delta=\Delta|_{r=1,\rho=1,\gamma=\mu^{-1/2}}$:

$$\bar\Delta = C(\mu^{1/2}h|\log h|)^{1/2}+C\mu^{-1/2}. \tag{3.103}$$

Then (3.102) becomes $O\bigl(\mu^{-1/2}h^{-3}+h^{-3}(\mu^{1/2}h|\log h|)^{1/2}\bigr)$.

One can check easily that the contribution of subzone $\bigl\{\bar\rho_2\ge\rho\ge C(\mu h|\log h|)^{1/2}\bigr\}$ is just $O(\mu h^{-3}\gamma^2\bar\rho_2^2)$ which is less than this. ◻

3.3.4 Again I would like to improve the remainder estimate achieved in proposition 3.19 under non-degeneracy condition $(0.8)_q$ with $q\ge 1$ also getting rid off assumption $|Z_1|\ge C(\mu h|\log h|)^{1/2}$. In the analysis below instead of W_n I consider its pullback $\omega(V-(2n+1)f_2\mu h)$ in the rescaled coordinates x' on Σ; it is useful for Hessians but it transforms into properties of W_n since I am interested only in the measures of sets $|\nabla W_n|\asymp\lambda$.

Proposition 3.20. *As $\mu\le ch^{-1}|\log h|^{-K}$ the contribution of $\mathcal{Z}_{\mathsf{inn},I}\cap\{|Z_1|\ge Cr\}$ to the remainder does not exceed $C\mu^{-1/2}h^{-3}$.*

Proof. Let us introduce

$$\ell=\ell_n=\epsilon r^{-1}|\nabla_{x_3,x_4,\xi_4}W_n|+\bar\ell, \tag{3.104}$$

and

$$\varrho=\varrho_n=\epsilon|\xi_3-k^*W_n^{1/2}|+\Delta. \tag{3.105}$$

Then the contribution of (r,ρ,ℓ)-elements to the asymptotics does not exceed (3.69)-like expression

$$C\mu h^{-3}\gamma r^2(\rho^2+r\ell^2)\times\Bigl(C_0(\mu h)^{-1}(\rho+r\ell^2)+1\Bigr); \tag{3.106}$$

if $\rho^2+r\ell^2\le C\mu h\gamma|\log h|$ then also $r^2\le C\mu h\gamma|\log h|$ and (3.106 does not exceed $C\mu^3h^{-1}\gamma^3|\log h|^2\le C\mu^{3/2}h^{-1}|\log h|^2$. So let us consider $\rho^2+r\ell^2\ge C\mu h\gamma|\log h|$.

As $\rho^2\ge Cr$ then periodicity is broken for sure (since $\nabla W_n|\ge\epsilon\rho^2-Cr$ and one can instantly upgrade $T_0=Ch|\log h|(\rho^2+r\ell^2)^{-1}$ to $T_2=\epsilon$. Otherwise it could be done as $\ell\ge C(\mu h|\log h|)^{1/2}$. Then contribution of such elements to the remainder does not exceed

$$\begin{aligned}C\mu h^{-3}\gamma r^2(\rho^2+r\ell^2)\times\Bigl(C_0(\mu h)^{-1}(\rho+r\ell^2)+1\Bigr)\times h|\log h|(\rho^2+r\ell^2)^{-1}=\\ C\mu h^{-2}\gamma r^2\times\Bigl(C_0(\mu h)^{-1}(\rho+r\ell^2)+1\Bigr)|\log h|\end{aligned} \tag{3.107}$$

because $|\operatorname{Hess} W_n|\geq \epsilon\rho^2 - Cr^2$ and one can delete safely logarithmic factor because it is not in the estimate of the Fourier transform.

Otherwise one can take $T_1=\epsilon\mu^{-1}\gamma^{-1}$ and then one gets

$$C\mu h^{-3}\gamma r^2(\rho^2+r\ell^2)\times\Bigl(C_0(\mu h)^{-1}(\rho+r\ell^2)+1\Bigr)\times h|\log h|(\rho^2+r\ell^2)^{-1}\mu\gamma\times\ell\Delta = C\mu^2h^{-2}\gamma^2r^2\times\Bigl(C_0(\mu h)^{-1}(\rho+r\ell^2)+1\Bigr)\bar\ell\Delta|\log h|\leq Ch^{-3}\times(\mu^{1/2}h) = C\mu^{1/2}h^{-2}|\log h|^2 \tag{3.108}$$

where one copy of $(\mu^{1/2}h|\log h|)^{1/2}$ comes as the width of zone with respect to ξ_3 and another as the width with respect to $|\nabla W_n|$. $\square$

Proposition 3.21. *Under condition* $(0.8)_1$ *be fulfilled. Then*
(i) The total contribution of $\mathcal{Z}_{\mathsf{inn},I}\cap\{|Z_1|\leq\epsilon r\}$ *to the remainder does not exceed* $C\mu^{-1/2}h^{-3}$ *as* $\mu\leq h^{-1}|\log h|^{-K}$.
(ii) The total contribution of $\mathcal{Z}_{\mathsf{inn},I}\cap\{|Z_1|\geq\epsilon r\}$ *to the remainder does not exceed*

$$C\mu^{-1/2}h^{-3}+C\mu^{5/4}h^{-3/2}|\log h|^{1/2}; \tag{3.109}$$

in particular it is $O(\mu^{-1/2}h^{-3})$ *as* $\mu\leq h^{-6/7}|\log h|^{-2/7}$ [28].

Again the main part of this asymptotics is given by expression (0.12).

Proof. Note that the most delicate is the case $\epsilon r\leq\rho\leq cr$ when even condition $(0.8)_2$ does not yield $(0.8)_1$ for W_n [29]

(a) Consider first $r\leq\epsilon_1$, $\rho\asymp r$, $\gamma\asymp\mu^{-1/2}r^{1/2}\rho^{1/2}\asymp\mu^{-1/2}r$.

Let us introduce a scaling function

$$\ell_n=\epsilon r^{-1}|\nabla W_n|+\frac{1}{2}\bar\ell,\qquad \bar\ell=C(\mu^{1/2}h|\log h|)^{1/2} \tag{3.110}$$

and the corresponding subpartition as $\rho\leq cr$. Note that actually $\ell\geq\bar\ell=C(\mu\gamma r^{-1}h|\log h|)^{1/2}$ is the condition that for time $T_1=\epsilon\mu^{-1}\gamma^{-1}$ the shift is observable. Then exactly as before

(3.111) The total contribution to the remainder of all ℓ-elements with $\ell\geq\bar\ell$, $\rho\asymp r$ does not exceed $C\mu^{-1/2}h^{-3}$.

---

[28] Actually I am going to prove later that one can drop logarithmic factors. In this case estimate cannot be improved without either additional assumptions or correction terms.

[29] However $(0.8)_1$ with the first and the second order derivatives taken only with respect to Λ implies such condition for W_n thus eliminating vicinity of Λ from the contributing more than $C\mu^{-/2}h^{-3}$ to the remainder.

So only $\ell \le \bar{\ell}$ subelements should be considered. Note that after rescaling $|\nabla\omega f_2| \asymp 1$ and therefore

(3.112) For each $\bar{\ell}$ subelement there exists $\bar{n}$ such that $|\nabla\omega(V-(2n+1)f_2\mu h| \ge \bar{\ell}$ as $|n-\bar{n}| \ge C\bar{\ell}(\mu h)^{-1}$ and therefore over each $\bar{\ell}$-element there are no more than $M = C\bar{\ell}(\mu h)^{-1}$ other subelements.

Therefore the total contribution of them to the remainder does not exceed

$$C\mu h^{-3}\gamma r^2(\rho^2+r\ell^2)\times\Bigl(C_0(\mu h)^{-1}\bar{\ell}+1\Bigr)\times h(\rho^2+r\ell^2)^{-1}\times\mu\gamma\Delta \asymp C\mu h^{-2}r^3\times\Bigl(\bar{\ell}^2(\mu h)^{-1}+\bar{\ell}\Bigr) \tag{3.113}$$

since $\Delta \asymp \bar{\ell}$. After summation over r-partition one gets $C\mu^{1/2}h^{-3}|\log h| + C\mu h^{-2}\bar{\ell}$ which is exactly (3.109).

(b) Consider now $r \asymp 1$, $\rho \asymp 1$, $\gamma \asymp \mu^{-1/2}$. One needs to recover an extra factor μh in comparison with estimate of proposition 3.19 and only case $\mu \ge h^{\delta-2/3}$ needs to be considered.

Let us assume first that

$$|\operatorname{Hess}(\frac{V}{f_2})| \ge \epsilon_0 \tag{3.114}$$

and let us fix

$$\zeta \ge \bar{\zeta} = C\bigl(\mu^{1/2}h|\log h|\bigr)^{\frac{1}{3}} + \epsilon\mu h \tag{3.115}$$

and examine subzone

$$\Omega_\zeta \overset{\mathrm{def}}{=} \bigl\{\exists n\in\mathbb{Z}^+ :\ |\operatorname{Hess} W_n| \asymp \zeta\bigr\}. \tag{3.116}$$

Let us introduce ζ-admissible partition in this zone; as $r \le \zeta$ one should replace it by r-admissible partition. Note that one can assume that

(3.117) For each element of such partition no more than $M \overset{\mathrm{def}}{=} C_0\zeta(\mu h)^{-1}$ magnetic numbers n satisfy condition $|\operatorname{Hess} W_n| \asymp \zeta$.

Really, if $|\nabla\omega f_2| \asymp 1$ then arguments of (a) work. If $|\nabla\omega f_2| \asymp\le \epsilon_1$ but $|\nabla f_2^{-1}V| \asymp 1$ then $|\nabla\omega V| \asymp 1$ and then $|\nabla W_n| \asymp 1$ for all $n \le C(\mu h)^{-1}$ and everything is easy. So, let both $|\nabla\omega f_2| \asymp\le \epsilon_1$ and $|\nabla f_2^{-1}V| \le \epsilon_1$. Then as $|\operatorname{Hess}\omega f_2| \asymp 1$ then (3.117) is obviously true.

Finally, if $|\operatorname{Hess}\omega f_2|\le \epsilon_1$ then (3.114) implies that $|\operatorname{Hess}\omega V|\asymp 1$ and then $|\operatorname{Hess} W_n|\asymp 1$ for all $n\le C(\mu h)^{-1}$.

Then for each index n described (3.117) all the above arguments could be repeated but with

$$\ell=\zeta^{-1}|\nabla W_n|+\frac{1}{2}\bar{\ell},\qquad \bar{\ell}\stackrel{\mathrm{def}}{=}C(\zeta^{-1}\mu h|\log h|)^{1/2}\tag{3.118}$$

leading to the contribution to the remainder

$$C\mu^2h^{-3}\gamma^2\times\zeta(\mu h)^{-1}\times h|\log h|\times\bar{\ell}\Delta\asymp C\mu^{1/2}h^{-2}|\log h|^2\tag{3.119}$$

since $\zeta\bar{\ell}=\Delta$.

So I am left with $\zeta=\bar{\zeta}$ and only with indices n $|\operatorname{Hess} W_n|\le\bar{\zeta}$. Here one needs to consider only subelements with $\ell\le\bar{\ell}$ i.e. with

$$|\nabla W_n\le C\bar{\zeta}^2.\tag{3.120}$$

So, I am looking on $\bar{\zeta}$ partition elements satisfying this condition.

Assume first that $|\operatorname{Hess}\omega f_2|\asymp 1$. Let us introduce scaling function

$$\ell=|\nabla\omega f_2|+\frac{1}{2}\bar{\ell};\tag{3.121}$$

then as $\ell\ge\bar{\ell}$ over each such subelement leave no more than $C\bar{\zeta}^2/(\mu h\ell)$ indices n satisfying (3.121) and then (3.119) is replaced by

$$C\mu^2h^{-3}\gamma^2\times\bar{\zeta}^2\ell^{-1}(\mu h)^{-1}\times h|\log h|\times\Delta\asymp Ch^{-3}\Delta\bar{\zeta}^2\ell^{-1}|\log h|;\tag{3.122}$$

alternatively one can replace (3.119) by

$$C\mu^2h^{-3}\gamma^2\times\bar{\zeta}(\mu h)^{-1}\times h|\log h|\times\ell\Delta\asymp Ch^{-3}\Delta\bar{\zeta}\ell|\log h|\tag{3.123}$$

where ℓ is a width of of the strip where $|\nabla\omega f_2|\le\ell$.

Comparing (3.122) and (3.123) one can see that the best choice of $\bar{\ell}$ is $\bar{\ell}=\bar{\zeta}^{1/2}$; then both of them become $Ch^{-3}\Delta\bar{\zeta}^{3/2}|\log h|\asymp C\mu^{1/2}h^{-3}|\log h|^2$.

On the other hand, as $|\operatorname{Hess}\omega f_2|\le\epsilon_1$ then due to the arguments proving (3.117) this analysis is not needed.

On the other hand, if (3.114) is replaced by condition

$$|\nabla(\frac{V}{f_2})|\ge\epsilon_0\tag{3.124}$$

the analysis is essentially the same, but simpler, with (3.116)–(3.117) replaced by

$$\zeta \geq \bar{\zeta} = c(\mu^{1/2}h|\log h|)^{\frac{1}{2}} + \mu h, \tag{3.125}$$

$$\Omega_\zeta \overset{\mathsf{def}}{=} \{\exists n \in \mathbb{Z}^+ :\ |\nabla W_n| \asymp \zeta\}, \tag{3.126}$$

respectively and in (3.118) $\mathsf{Hess}\, W_n| \asymp \zeta$ is replaced by $|\nabla W_n| \asymp \zeta$.

(c) Finally as $\rho \leq \epsilon r$ condition $(0.8)_1$ translates for a similar condition to W_n and then one can apply the arguments similar to those of proposition 3.20 which imply (i). I leave easy details to the reader.

(d) Analysis of $\rho^2 \leq c(\mu h|\log h|)^{1/2}$ is also easy. Let us introduce ℓ by (3.110). Then as $\rho^2 + r\ell^2 \leq C\mu^{1/2}h|\log h|$ factor $(\rho^2 + r\ell^2)$ takes care of everything; otherwise factor Δ is retained but factor $(\rho^2 + r\ell^2)$ translates in the end into extra factor $\mu h|\log h$. I leave easy details to the reader. □

3.3.5 Now the better remainder estimate should be pursued only as $\mu \geq h^{-6/7}|\log h|^{-2/7}$. Further, as $q = 0, 1$ the contribution of $\mathcal{Z}_{\mathsf{per}}$ to the remainder is better than the remainder estimates $O(\mu^{-1/2}h^{-3} + \mu^2 h^{-2})$, $O(\mu^{-1/2}h^{-3} + \mu^{3/2}h^{-2-\delta})$ and therefore only $q = 2, 3$ should be considered.

Moreover, it follows from the proof of proposition 3.21 that if summation is taken over $\mathcal{Z}_{\mathsf{per}} \cap \{\mathsf{dist}(x, \Lambda) \leq r\}$ then the second term in (3.109) would get an extra factor r^4 (from $r^2\gamma^2$) and therefore only zone $\mathcal{Z}_{\mathsf{per}} \cap \{\mathsf{dist}(x, \Lambda) \geq \bar{r} = \mu^{-1/16}|\log h|^{-1/8}\}$ should be considered.

Moreover, then $\bar{\zeta} \asymp \mu h$ (which is much larger than the first term in (3.114) and therefore for each $\bar{\zeta}$-partition subelement only one magnetic number n should be considered.

There is a minor problem as $\mu \geq h^{-1}|\log h|^{-K}$ because then contributions of $\{\rho \leq \epsilon r\}$ and $\{\rho \geq Cr\}$ were estimated by $\mu^{1/2}h^{-2}|\log h|^K$; however in this case there also factor r^2 at least coming if looking at $\mathcal{Z}_{\mathsf{per}} \cap \{\mathsf{dist}(x, \Lambda) \leq r\}$ and therefore one needs to consider only $\mathcal{Z}_{\mathsf{per}} \cap \{\mathsf{dist}(x, \Lambda) \geq |\log h|^{-K}\}$. Then in the first case condition $(0.8)_2$ supplies factor $(\mu^{1/2}h|\log h)^{1/2}|\log h|^K$ which is more than enough to take care of $|\log h|^K$; however I will just include both cases in the final analysis,

Then repeating all arguments of subsections 2.8, 2.9 of [Ivr6] with the obvious modifications, due to the presence of r (which is greater than $\bar{r} \geq \bar{r}$) and of $(x_2, \mu^{-1}hD_2)$ in all the operators, one can prove easily the following

Proposition 3.22. *Under condition $(0.8)_2$ the total contribution of $\mathcal{Z}_{\mathsf{inn},l} \cap \{|Z_1| \geq \epsilon\}$ to the remainder does not exceed $C\mu^{-1/2}h^{-3}$ while the main part of the asymptotics is given*

by (*0.12*) *plus a correction term*

$$\int \mathcal{E}^{\mathsf{MW}}_{\mathsf{corr}}(x',\tau)\psi(x')\,dx', \tag{3.127}$$

where temporarily

$$\mathcal{E}^{\mathsf{MW}}_{\mathsf{corr}} \overset{\mathsf{def}}{=} h^{-1}\int\int_{-\infty}^{0}\Bigl(F_{t\to h^{-1}\tau}\bigl(\bar\chi_T(t)-\bar\chi_{T_0}(t)\bigr)\bigl(\Gamma_x(uQ_y^t)\bigr)\Bigr)\,d\tau dx_1 \tag{3.128}$$

and in this formula one can choose arbitrarily

$$T_0\in[Ch|\log h|,\epsilon'\mu^{-1/2}\gamma^{-1}], \tag{3.129}$$

$$T\ge Cr^{-1/2}(h|\log h|)^{1/2}\bigl(\max(\bigl|\xi_3-k^*|Z_1|\bigr|,h^{1/2})\bigr)^{-1} \tag{3.130}$$

and pseudo-differential $Q\equiv I$ *in* $\mathcal{Z}_{\mathsf{per}}\cap\{\mathsf{dist}(x,\Lambda)\ge\bar r\}$ *and supported in the same zone, slightly inflated.*

In the next section this correction term will be rewritten in more explicit form.

3.3.6 Conclusion. So, with the remainder estimates described in theorems 0.1, 0.3 asymptotics are derived where the main parts are given by rather implicit formula (4.1).

The analysis in frames of theorem 0.5 is similar but much simpler. I leave it to the reader.

4 Calculations

So, according to the previous section, with the remainder estimates described there the main part of the asymptotics of $\int e(x,x,0)\psi(x)\,dx$ is given by

$$\sum_\iota h^{-1}\int_{-\infty}^{0}\Bigl(F_{t\to h^{-1}\tau}\bar\chi_{T_\iota}(t)\Gamma\bigl(u\psi Q_{\iota y}^t\bigr)\Bigr)\,d\tau \tag{4.1}$$

where Q_ι form an appropriate partition of unity and $T_\iota\ge T_{(\iota)0}$ also described there.

In this section I will make all the calculations transforming implicit expression (4.1) to a more explicit form, namely to

$$\int \mathcal{E}^{\mathsf{MW}}(x,0)\psi(x)\,dx. \tag{4.2}$$

There are many methods to make a reduction and I will employ them depending on the power of the magnetic field and non-degeneracy condition.

Also I will calculate (3.122) in a more explicit form.

4.1 Moderate magnetic field. I

I remind that I proved that for $\mu \le h^{-\delta}$ the remainder estimate $O(\mu^{-1/2}h^{-3})$ holds while the main part is given by (0.12) with $T = \bar{T} \overset{\text{def}}{=} Ch|\log h|$. Now I want to prove the same under a larger upper bound for μ; further, increasing μ I want to recover a remainder estimate consistent with those in theorems 0.1, 0.3, 0.5 while keeping the main part given by (0.12) with $T = \bar{T}$. To do this I need to estimate properly (1.22)-like correction term

$$h^{-1}\int_{-\infty}^{0}\Bigl(F_{t\to h^{-1}\tau}\bigl(\bar{\chi}_{T'}(t) - \bar{\chi}_{T''}(t)\bigr)\bigl(\Gamma uQ_y^t\bigr)\Bigr)\,d\tau \tag{4.3}$$

with $T' = T_0$ derived in section 3 and $T'' = \bar{T}$ [30]. This will let me to rewrite (0.12) as a Weyl expression, may be with $O(\mu^2h^{-2})$ correction. Here $Q = \psi$ but in the analysis below I will need more general PDOs.

This expression (4.3) is the sum of expressions of the type

$$h^{-1}\int_{-\infty}^{0}\Bigl(F_{t\to h^{-1}\tau}\bigl(\bar{\chi}_{2T}(t) - \bar{\chi}_{T}(t)\bigr)\Gamma uQ_y^t\Bigr)\,d\tau = iT^{-1}\Bigl(F_{t\to h^{-1}\tau}\chi_T(t)\Gamma uQ_y^t\Bigr)\Big|_{\tau=0} \tag{4.4}$$

with $\chi(t) = t^{-1}\bigl(\bar{\chi}(\frac{1}{2}t) - \bar{\chi}(t)\bigr)$ and $T = 2^kT''$, k running from 2 to $\lceil \log_2(T'/T'')\rceil$.

Proposition 4.1. *Let assumptions of theorem 0.1 with*

$$h^{-\delta} \le \mu \le \epsilon(h|\log h|)^{-1} \tag{4.5}$$

be fulfilled and let Q be supported in $\{\mathsf{dist}(x,\Sigma) \le \gamma\}$, $\gamma \ge \mu^{-1/2}$.

Then expression (4.3) with $T' = \epsilon\mu^{-1}\gamma^{-1}$ and $T'' = \bar{T}$ does not exceed $C\mu^{-1/2}h^{-3} + C\mu^2h^{-2}$ [31].

Proof. Note first that expression (4.4) with $T \in [\bar{T}, \epsilon\mu^{-1}]$ is negligible due to condition (0.2). So, until the end of the proof I can consider $T'' = \epsilon\mu^{-1}$.

Further note that, as Q is supported in $\{|Z_1|^2 \ge C\mu h|\log h|\}$ expression (4.4) with $T \in [T'', T']$ is negligible as well. Then for a general PDO Q expression (4.4) does not exceed $C\mu^2h^{-2}|\log h|$.

Then proposition is proven as $\mu \le c(h|\log h|)^{-2/5}$ and until the end of the proof I can assume that

$$(h|\log h|)^{2/5} \le \mu \le \epsilon(h|\log h|)^{-1}. \tag{4.6}$$

[30] However in what follows I will consider some other parameters $T' > T''$.

[31] Term $C\mu^{-1/2}h^{-3}$ is probably of no need in some estimates here and below but it is in the remainder estimate anyway.

Further, as Q is supported in $\{\mathsf{dist}(x,\Sigma)\le\bar\gamma_3=c|\log h|^{-1}\}$, expression (4.3) would not exceed $C\mu^2h^{-2}\bar\gamma_3|\log h|\asymp \mu^2h^{-2}$ and as Q is supported in $\{|Z_1|^2\le c\mu h\}$ expression (4.3) would not exceed $C\mu^2h^{-2}$ as well.

So, the only zone in question is

$$\bigl\{C_0\mu h\le\rho^2\le C\mu h|\log h|\bigr\}\cap\bigl\{\mathsf{dist}(x,\Sigma)\asymp\gamma, |Z_1|\asymp\rho\bigr\} \tag{4.7}$$

with $|\log h|^{-1}\le\gamma\le\epsilon$. In this zone one needs to consider expression (4.4).

Using precanonical form and decomposition (2.16) one can rewrite (4.4) as the sum of the similar expressions but with u, Q,Γ replaced by u_n, Q_n and Γ' respectively:

$$iT^{-1}\Bigl(F_{t\to h^{-1}\tau}\chi_T(t)\Gamma' u_nQ^t_{ny}\Bigr)\Bigr|_{\tau=0}. \tag{4.8}$$

From the decomposed problem

$$(hD_t-\mathcal{A}_n)u_{nn'}\equiv 0, \tag{4.9}$$

$$u_{nn'}|_{t=0}\equiv\delta_{nn'}\delta(x'-y') \tag{4.10}$$

one can prove not only that expression (4.8) is negligible as $T\ge T_\rho\overset{\mathsf{def}}{=}Ch\rho^{-2}|\log h|$ but also that its absolute value does not exceed $C\mu h^{-2}\gamma\rho^2\bigl(h\rho^{-2}T\bigr)^{-s}$ as $t\le T_\rho$. The summation of this expression with respect to T from $\epsilon\mu^{-1}$ to ∞ results in $C\mu h^{-2}\gamma\rho^2\bigl(\mu h\rho^{-2}\bigr)^{-s}$.

Further, summation of the latter expression with respect to n results in

$$C\mu h^{-2}\gamma\rho^2\bigl(\mu h\rho^{-2}\bigr)^{-s}\times\rho^2(\mu h)^{-1}=Ch^{-3}\rho^4\bigl(\mu h\rho^{-2}\bigr)^{-s}$$

since no more than $\rho^2(\mu h)^{-1}$ numbers n violate the ellipticity of $\mathcal{A}_n$.

Finally, the summation with respect to $\rho\ge c(\mu h)^{1/2}$ results in $C\mu^2h^{-2}\gamma$. So, one can conclude that expression (4.3) with indicated T' and T'' by absolute value does not exceed $C\mu^2h^{-2}$. $\square$

One can improve the above error estimate under nondegeneracy condition.

Proposition 4.2. *Let assumptions of theorem 0.1 and condition (4.6) be fulfilled. Further, let Q be supported in $\{\mathsf{dist}(x,\Sigma)\le\gamma\}$ with $\gamma\ge\mu^{\delta-1/2}$. Finally, let condition $(0.9)_q$ with $q=1,2$ be fulfilled.*

Then expression (4.3) with $T'=\epsilon\mu^{-1}\gamma^{-1}$, $T''=\bar T$ and Q supported in $\{\mathsf{dist}(x,\Sigma)\le\gamma\}$ does not exceed

$$C\mu^{-1/2}h^{-3}+C(\mu h)^{2+q/2}|\log h|^Kh^{-4}\gamma. \tag*{$(4.11)_q$}$$

Proof. There is no need to consider the inner core. Further, due to proposition 4.1 the case $\mu \le Ch^{-2/5}$ is already covered. So, let us assume that $\mu \ge h^{-2/5}$.

Consider precanonical form (2.17) and define $\ell = |\nabla^{\#}_V$, $\rho = |Z^{\#}_1|$. Then even as $\rho \le \bar{\rho} \overset{\mathsf{def}}{=} C(\mu h|\log h|)^{1/2}$ but $\ell \ge \bar{\ell} \overset{\mathsf{def}}{=} \bar{\rho}$ one can still trade $T' = \epsilon\mu^{-1}\gamma^{-1}$ to $T'' = \epsilon\mu^{-1}$ because then the shift for time $T \in [\epsilon\mu^{-1}, T'']$ is still observable. Here perturbation terms and their derivatives are at most $C\mu^{-2} \ll \bar{\ell}$.

On the other hand, the total contribution to the remainder of all partition elements with $\rho \le \bar{\rho}$, $\ell \le \bar{\ell}$, $\mathsf{dist}(x,\Sigma) \le \gamma$ does not exceed $C\bar{\rho}^2\bar{\ell}^{2+q}\gamma h^{-4}$ which is exactly $(4.11)_q$. $\square$

Remark 4.3. While the error estimate of proposition 4.1 is as good as I need in the general case, the error estimate of proposition 4.2 is as good as I need under condition $(0.9)_q$ only under some restrictions to μ and/or γ.

4.2 Moderate magnetic field. II

Now I want to trade $T'' = \epsilon\mu^{-1}\gamma^{-1}$ to a larger value T_0 which provides the remainder estimate derived in section 3. Now I will need to use some canonical forms and thus different zones will be treated separately.

4.2.1 I will start from the strictly outer zone.

Proposition 4.4. *Let assumptions of theorem 0.1 and condition (4.5) be fulfilled and let Q be supported in $\mathcal{Z}^*_{\mathsf{out}} \cap \{\mathsf{dist}(x,\Sigma) \le \gamma\}$ with $\gamma \ge C_0\mu^{\delta-1/2}$. Then*

(i) In the general case expression (4.3) with $T' = T_0$ and $T'' = \bar{T}$ does not exceed

$$C\mu^{-1/2}h^{-3} + C(\mu h)^{(4+q)/2}h^{-4}\gamma|\log h|^K + C(\mu h)^{(3+q)/2}h^{-4}\gamma^2|\log h|^K \tag{4.12$_q$}$$

with $q = 0$;

(ii) Under condition $(0.9)_q$ this expression does not exceed $(4.12)_q$.

Proof. One needs to consider only Q supported in a zone in which in the proofs of propositions 4.1, 4.2 term (4.3) with $T' = \epsilon\mu^{-1}$ and $T'' = \bar{T}$ was negligible. Then I am going to prove $(4.12)_q$ without the second term.

(i) Assume first that Q is supported in $\mathcal{Z}_{(r,\gamma)} \subset \mathcal{Z}^*_{\mathsf{out},I}$.

Let us consider canonical form (2.54) and introduce function ℓ^* by (3.38). If $\ell \ge 1$ (i.e. $p \ge Cr^{-1}(\mu h)^{-1}$) I do not partition the "final" ball $B(0,1)$ corresponding to the original $(\gamma;\gamma r^{-1})$ element; otherwise I make ℓ-subpartition. Then on any given ℓ subpartition element the propagation speed with respect to x_3 does not exceed $\mu^{-1}\gamma^{-2}r\ell_p$; further the

propagation speed is of this magnitude as $|\partial_{\xi_3} H_{pn}| \asymp \ell$; in this case the shift for time T is observable as it satisfies the logarithmic uncertainty principle

$$\mu^{-1}\gamma^{-2}\ell \times T \times \ell \geq Cr^2\mu^{-1}h\gamma^{-3}|\log h|.$$

Plugging $T \asymp \mu^{-1}\gamma^{-1}$ one arrives to

$$\ell \geq \bar{\ell}_1 \overset{\text{def}}{=} C\bigl(\mu h|\log h|\bigr)^{1/2}. \tag{4.13}$$

Similarly, considering propagations with respect to other variables one arrives to the same conclusion as $|\nabla W_{np}| \asymp \ell$.

Note that for given index p and for index n violating ellipticity $\ell \asymp \ell_p = \bigl(\ell^* + r|p - \bar{p}|\mu h\bigr)$ with some $\bar{p} \leq C_0(\mu h)^{-1}$.

Further, on each ℓ^* subpartition element condition (4.13) is fulfilled for all indices p as $\ell^* \geq \bar{\ell}_1$ and condition (4.13) is violated for no more than $C_0\bar{\ell}_1(r\mu h)^{-1}$ ($\gg 1$) indices p as $\ell^* \leq \bar{\ell}_1$. Furthermore for each such index p ellipticity is violated by no more than $C_0\bigl(\bar{\ell}_1^2\gamma(r\mu h)^{-1} + 1\bigr)$ indices n.

Therefore the total contribution to (4.3) of all singular subelements (i.e. subelements with $\ell^* \leq C_0\bar{\ell}_1$) belonging to $\mathcal{Z}_{(r,\gamma)}$ does not exceed

$$C\mu^2 h^{-2}\gamma^2 r^2 \times \Bigl(\bar{\ell}_1(r\mu h)^{-1} + 1\Bigr) \times \Bigl(\bar{\ell}_1^2\gamma(r\mu h)^{-1} + 1\Bigr) \times \ell_1^q; \tag{4.14}$$

since the second and the third factors do not exceed $C\bar{\ell}_1(r\mu h)^{-1}$ and $C|\log h|^K$ respectively, I get $Ch^{-4}\bigl(\mu h\bigr)^{(3+q)/2}\gamma^2|\log h|^K$. This expression sums with respect to r,γ to the same expression calculated at the maximal values of γ and $r = 1$, and multiplied by $|\log h|$, which is exactly expression $(4.12)_q$. I increase K as needed.

(ii) Zone $\mathcal{Z}^*_{\mathsf{out},II}$ is treated in the same way. $\square$

While estimates $(4.11)_q$ with $q \geq 0$ and even $(4.12)_q$ with $q \geq 1$ are sufficient for my needs, $(4.12)_0$ is not and needs to be improved to $(4.12)_q$ with q arbitrarily close to 1:

Proposition 4.5. *Let assumptions of theorem 0.1 and condition (4.5) be fulfilled and let Q be supported in $\mathcal{Z}^*_{\mathsf{out}} \cap \{\mathsf{dist}(x,\Sigma) \leq \gamma\}$ with $\gamma \geq C_0\mu^{\delta-1/2}$. Then expression (4.3) with $T' = T_0$ and $T'' = \bar{T}$ does not exceed*

$$C\mu^{-1/2}h^{-3} + C\mu^2h^{-2}\gamma|\log h| + C(\mu h)^{(3+q)/2}h^{-4}\gamma^2|\log h|^K \tag{4.15}$$

with $q < 1$ arbitrarily close to 1.

Proof. Again, one needs to consider only Q supported in a zone $\{|Z_1| \ge C(\mu h|\log h|)^{1/2}\}$ in which in the proof of proposition 4.1 term (4.3) with $T' = \epsilon\mu^{-1}$ and $T'' = \bar{T}$ was negligible. Then I am going to prove (4.15) without the second term.

(i) Assume first that Q is supported in $\mathcal{Z}_{(r,\gamma)} \subset \mathcal{Z}^*_{\mathsf{out},I}$. One needs to consider only $p \le C_0(r\mu h)^{-1}$.

Let us introduce the scaling function and ℓ-admissible partition the in the "final" ball $B(0,1)$ corresponding to the original $(\gamma;\gamma r^{-1})$ element:

$$\ell^*_k = \ell_k \overset{\mathsf{def}}{=} \min_{n,p} \epsilon\Bigl(\zeta^{-1}\sum_{j\le k} |\nabla^j \mathcal{A}_{np}|^{(m+1)/(k+1-j)}\Bigr)^{1/(m+1)} + \bar{\ell}_k \tag{4.16}$$

with $k = m$, $\zeta = \gamma r^{-1}$ and

$$\bar{\ell}_k = \bigl(\mu h|\log h|^K\bigr)^{1/(k+1)}. \tag{4.17}$$

As $m = 1$ this function coincides with ℓ^* given by (3.38).

Consider first partition m-*singular* groups $((p,n),\mathsf{element})$ i.e. groups such that

$$|\partial^\alpha \mathcal{A}_{np}| \le c\zeta\ell^{k-|\alpha|} \qquad \forall\alpha : |\alpha| \le k \tag{4.18}$$

with $k = m$, $\ell = \bar{\ell}_m$ and $\zeta = \zeta_m$.

Then contribution of all m-singular groups in $\mathcal{Z}_{(r,\gamma)}$ could be estimated in the same manner as (4.14):

$$C\mu^2 h^{-2} r^2\gamma^2 \times \Bigl(\zeta\ell^k(\gamma\mu h)^{-1} + 1\Bigr) \times \Bigl(\zeta\ell^{k+1}(\mu h)^{-1} + 1\Bigr) \tag{4.19}$$

with the same k, ℓ and ζ as in (4.18). Here the second factor estimate the number of indices p involved and the third factor indicates the number of indices n violating ellipticity of $\mathcal{A}_{pn}$ for given p.

The groups which are not k-singular are k-*regular*, i.e. they satisfy (4.18) with some $\ell \ge 2\bar{\ell}_k$ and also

$$|\partial^\alpha \mathcal{A}_{np}| \asymp \zeta_k\ell^{k-|\alpha|} \qquad \text{for some } \alpha : |\alpha| \le k. \tag{4.20}$$

Then one can rescale such element to $B(0,1)$. After this rescaling conditions (4.18),(4.20) are fulfilled with $\ell \overset{\mathsf{redef}}{=} 1$ and $\zeta \overset{\mathsf{redef}}{=} \zeta\ell^k$. So, one can apply the same rescaling and partition to it with $(k-1)$ instead of k and with crucial ℓ (separating $(k-1)$-singular from $(k-1)$-regular) equal to $\bar{\ell}_{k-1}/\ell_k$. This means that if one returns to the original element, the radius of the k-singular element would be $\bar{\ell}_k$ every time.

This continuation however has a new property: every time when regular or singular ℓ'-subelements appear inside of some regular elements, their relative densities do not exceed ℓ'/ℓ (all the time I refer to the same scale) and therefore the absolute density of all ℓ'-subelements does not exceed $C\ell'\bar{\ell}_m^{-1}$.

Therefore the total contribution of all k-singular elements does not exceed (4.19) multiplied by $\bar{\ell}_k\bar{\ell}_m^{-1}$

$$C\bar{\ell}_m^{-1}\mu^2h^{-2}r^2\gamma^2 \times \Bigl(\zeta\ell^{k+1}(\gamma\mu h)^{-1}+\ell\Bigr)\times\Bigl(\zeta\ell^{k+1}(\mu h)^{-1}+1\Bigr) \tag{4.21}$$

with $\ell=\bar{\ell}_k$ and $\zeta=\gamma r^{-1}$ since I know that ζ does not increase (even if it is different in the different elements). Then two last factors do not exceed $|\log h|^K$ for sure and the total contribution of all singular $((p,n),\mathsf{subelement})$ groups does not exceed $\bar{\ell}_m^{-1}\mu^2h^{-2}\gamma^2r^2|\log h|^K$ and summation with respect to (r,γ) results in the same expression with the maximal values of γ and $r=1$.

Microlocally this is sound as logarithmic uncertainty principle

$$\bar{\ell}_k^2\ge\mu^{-1}hr^2\gamma^{-3}|\log h|^K \tag{4.22}$$

and to satisfy it for any $\gamma\gg\mu^{-1/2}r^{1/2}$ one needs to take $k=1$.

This leaves me with 1-regular groups and there are three types of them:

(a) With $\zeta\ell^2\le\gamma r^{-1}\mu h|\log h|^K$, which are covered by the same estimate (4.21);

(b) With $\zeta\ell^2\ge\gamma r^{-1}\mu h|\log h|^K$; for them the shift $\mu^{-1}\gamma^{-2}r\zeta\ell T$ with $T=\mu^{-1}\gamma^{-1}$ is observable since $\mu^{-1}r^2\gamma^{-3}\zeta\ell\times\mu^{-1}\gamma^{-1}\times\ell\ge Cr^2\mu^{-1}h\gamma^{-3}$;

(c) 0-regular groups but then $\mathcal{A}_{pn}$ is just elliptic on the corresponding elements.

(ii) Zone $\mathcal{Z}^*_{\mathsf{out},II}$ is treated in the same way. □

4.2.2 In the general case my purpose is the remainder estimate $O(\mu^{-1/2}h^{-3}+\mu^2h^{-2})$.

Proposition 4.6. *Let assumptions of theorem 0.1 be fulfilled. Then*
(i) As

$$h^{-\delta}\le\mu\le\bar{\mu}_0\stackrel{\mathrm{def}}{=}=h^{\delta-2/5} \tag{4.23}$$

$\int e(x,x,\tau)\psi(x)\,dx$ *is given modulo* $O(\mu^{-1/2}h^{-3})$ *by expression* (*0.12*) *with* $T=\bar{T}$;

(ii) Moreover, as

$$h^{\delta-2/5} \le \mu \le h^{\delta-1} \tag{4.24}$$

and $\psi(x)$ *is a nice*[32] *function supported in* $\{|x_1| \le \bar{\gamma}_2 = \mu^{-\delta'}\}$ $\int e(x,x,\tau)\psi(x)\,dx$ *is given modulo* $O(\mu^{-1/2}h^{-3}+\mu^2h^{-3})$ *by expression* (0.12) *with* $T=\bar{T}$

Proof. Combining propositions 3.2, 3.4 and 4.5 I conclude that this estimate holds in frames of (i) for ψ supported in $\mathcal{Z}^*_{\mathsf{out}}$ and in frames of (ii) for ψ supported in $\mathcal{Z}^*_{\mathsf{out}} \cap \{|x_1| \le \mu^{-\delta}\}$.

In propositions 3.11, 3.14, 3.16 and 3.19 it was essentially proven that this estimate holds for ψ supported in the near outer zone $\mathcal{Z}'_{\mathsf{out}}$, inner core, inner bulk zone $\mathcal{Z}_{\mathsf{inn},II}$ and true inner zone $\mathcal{Z}_{\mathsf{inn},I}$ respectively. □

4.3 Intermediate magnetic field. I

4.3.1 Now I want just to calculate (0.12) with $T = Ch|\log h|$. It is well-known (see for example [Ivr5]) that

Proposition 4.7. *Let* $\mu \le h^{-1+\delta}$ *with arbitrarily small* $\delta > 0$*. Then under condition* (0.2)
(i) Asymptotics holds

$$h^{-1}\int_{-\infty}^{\lambda} \Bigl(F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma_x u Q_y^t\Bigr)\,d\tau \sim \sum_{(n,m)\in\mathbb{Z}^{+\,2}} \kappa_{n,m,Q}(x,\lambda)h^{-4+2m+2n}\mu^{2n} \tag{4.25}$$

as $|\lambda| \le \epsilon$, $\bar{T} \le T \le \epsilon\mu^{-1}$.
(ii) Moreover, as $Q = Q(x)$ $\quad \kappa_{n,0,Q}(x,.) = \kappa_{n,0,I}(x,.)Q(x)$ *and*

$$Th^{-1}\int_{-\infty}^{\infty} \hat{\bar{\chi}}\bigl((\lambda-\tau)Th^{-1}\bigr)\mathcal{E}^{\mathsf{MW}}(x,\tau)\,d\tau \sim \sum_{n\in\mathbb{Z}^+} \kappa_{n,0,I}(x,\lambda)h^{-4+2m+2n}\mu^{2n} \tag{4.26}$$

with the Standard Weyl Expression

$$\kappa_{0,0,I}(x,\lambda)h^{-4} = \mathcal{E}^{\mathsf{W}}(x,\tau) \stackrel{\mathrm{def}}{=} \frac{1}{32\pi^2}(2\tau+V)^2\sqrt{g}. \tag{4.27}$$

Combining propositions 4.6(i) and 4.7(i) I arrive to

Corollary 4.8. *Under conditions* (0.2) *and* (4.23) *modified asymptotics* (0.3) *(with* $\mathcal{E}^{\mathsf{MW}}$ *replaced by* $\mathcal{E}^{\mathsf{W}}$*) holds.*

---

[32] F.e. $\psi(x) = \psi_1(x_1/\bar{\gamma}_2)\psi_2(x_2,x_3,x_4)$.

To prove theorem 0.1 under condition (4.23) it is sufficient to estimate properly

$$|\int \Bigl(\mathcal{E}^{\mathsf{MW}}(x,\lambda)-\mathcal{E}^{\mathsf{W}}(x,\lambda)\Bigr)\psi(x)\,d\tau| \tag{4.28}$$

which due to proposition 4.7(ii) is equivalent to

$$|\int \Bigl(\mathcal{E}^{\mathsf{MW}}(x,\lambda)-Th^{-1}\int_{-\infty}^{\infty}\widehat{\bar\chi}\bigl((\lambda-\tau)Th^{-1}\bigr)\mathcal{E}^{\mathsf{MW}}(x,\tau)\,d\tau\Bigr)\psi(x)\,d\tau|. \tag{4.29}$$

Proposition 4.9. *(i) Under conditions (0.2) and (4.23) both expressions (4.28), (4.29) with $\lambda=0$ do not exceed $C\mu^{-1/2}h^{-3}$.*

(ii) Under conditions (0.2) and (4.24) both expressions (4.28), (4.29) with $\lambda=0$ and a nice[32] function ψ supported in $\{|x_1|\le\bar\gamma_2=\mu^{-\delta'}\}$ do not exceed $C\mu^{-1/2}h^{-3}+C\mu^2h^{-2}$.

Proof. Proof is standard, based on the same scaling functions and partition as in the proof of proposition 4.5 as ψ is supported in $\mathcal{Z}^*_{\mathsf{out}}$. These arguments work in the other zones as well since I do not need uncertainty principle anymore. I leave the easy details to the reader. □

Now propositions 4.6(i), 4.7 and 4.9(i) imply

Corollary 4.10. *Under conditions (0.2) and (4.23) asymptotics (0.3) holds.*

4.3.2 In view of theorem 0.1 nondegeneracy condition should be used only as

$$ch^{\delta-2/5}\le\mu\le Ch^{-1} \tag{4.30}$$

with $\delta=0$. Then, under condition $(0.9)_1$ the remainder estimate derived in section 3 is $O(\mu^{-1/2}h^{-3}+\mu^{3/2}h^{-3/2-\delta})$ i.e. it is $O(\mu^{-1/2}h^{-3})$ as $\mu\le h^{\delta-3/4}$ and then there is no need to invoke $(0.9)_q$ with $q\ge 2$. However it makes life much easier and I will do it right now leaving more difficult analysis for the later; so $\delta>0$ is arbitrarily small in this subsubsection.

First of all one needs to prove

Proposition 4.11. *Let conditions (0.2) and $(0.9)_q$ be fulfilled and $\mu\le h^{\delta-1}$. Let ψ be a nice function supported in $\{|x_1|\le\gamma\}$ with $\mu^{1/2-\delta}\le\gamma\le\epsilon$. Then $\int e(x,x,\tau)\psi(x)\,dx$ is given by expression (0.12) with $T=\bar T$ with an error not exceeding $(4.12)_q$.*

Proof. Combining propositions 3.7 and 4.4 I conclude that the statement holds for ψ supported in $\mathcal{Z}^*_{\mathsf{out}}\cap\{|x_1|\le\gamma\}$.

In propositions 3.14, 3.12, 3.17 and 3.22 it was essentially proven that this estimate holds for ψ supported in the near outer zone $\mathcal{Z}'_{\mathsf{out}}$, inner core, inner bulk zone $\mathcal{Z}_{\mathsf{inn},II}$ and true inner zone $\mathcal{Z}_{\mathsf{inn},I}$ respectively. □

Proposition 4.9 and 4.7 imply

Corollary 4.12. *Let conditions* (0.2) *and* $(0.9)_q$ *be fulfilled and* $\mu \le h^{\delta-1}$. *Then*
(i) One can define the main part of asymptotics by (0.12) *with* $T = Ch|\log h|$ *(or equivalently by* (4.26) *while the the remainder does not exceed* $(4.12)_q$ *with* $\gamma = 1$;
(ii) In particular, the remainder is $O(\mu^{-1/2}h^{-3})$ *as*

$$\mu \le \bar{\mu}_q \stackrel{\mathsf{def}}{=} h^{-\frac{q+1}{q+4}}|\log h|^{-K} \tag{4.31$_q$}$$

under condition $(0.9)_q$ *with* $q \ge 1$; *in particular* $\bar{\mu}_2 = \mu^{-1/2}|\log h|^{-K}$ *and* $\bar{\mu}_3 = \mu^{-4/7}|\log h|^{-K}$.

I refer to the case of $(4.31)_q$ as a *moderate magnetic field*. While I can get rid off logarithmic factors, there is no point to do it right now.

To prove theorem 0.3 one needs to estimate properly expressions (4.28) and (4.29).

Proposition 4.13. *Let conditions* (0.2) *and* $(0.9)_q$ *be fulfilled and* $\mu \le h^{\delta-1}$. *Let* ψ *be a nice function supported in* $\{|x_1| \le \gamma\}$ *with* $\mu^{\delta-1/2} \le \gamma \le \epsilon$. *Then*
(i) Both expressions (4.28) *and* (4.29) *do not exceed* $(4.12)_q$.
(ii) In particular as $\gamma = 1$ *both expressions* (4.28) *and* (4.29) *do not exceed* $C\mu^{-1/2}h^{-3}$ *as* $\mu \le \mu_q$.

Proof. Proof is standard, based on the same scaling functions and partition as in the proof of proposition 4.4 as ψ is supported in $\mathcal{Z}^*_{\mathsf{out}}$. These arguments work in the other zones as well since I do not need uncertainty principle anymore. I leave the easy details to the reader. □

Corollary 4.14. *Theorem 0.3 is proven as* $\mu \le \mu_q$.

4.4 Intermediate magnetic field. II

4.4.1 Now I want to prove theorems 0.1, 0.3 as $\mu \ge \bar{\mu}_q$ defined by (4.23), $(4.31)_q$ for $q = 0$ and $q \ge 1$ respectively. To do this I need to consider expression (0.12) localized in this singular zone because in the regular zone one can always apply (0.12) with $Q = I$ even if the expression is not as explicit as in the case $Q = I$.

Proposition 4.15. *One can rewrite with the same error as a remainder estimate derived in section 3 expression* (0.12) *as*

$$\int \widehat{\mathcal{E}}^{\mathsf{MW}}_{\bar{Q}}(x,0)\psi(x)\,dx \tag{4.32}$$

with

$$\widehat{\mathcal{E}}^{\mathsf{MW}}_{\bar{Q}}(x,0) \stackrel{\mathrm{def}}{=} \mathrm{const}\sum_{n,p}\Bigl(\theta(\mathcal{A}_{pn})\bar{Q}_{pn}\Bigr)\Bigr|_{(x'',\xi'')=\Psi^{-1}(x)} \times f_1(x)f_2(x)\mu^2h^{-2} \tag{4.33}$$

with exactly the same constant as in $\mathcal{E}^{\mathsf{MW}}$.

Proof. Proof is rather standard: I consider canonical form of the operator and in this form I apply the method of successive approximations with operator in question $\mathcal{A}_{p,n}(x,\mu^{-1}hD_x)$ and unperturbed operator $\mathcal{A}_{p,n}(y,\mu^{-1}hD_x)$. I leave details repeating those in my multiple papers to the reader. $\square$

Remark 4.16. Obviously, defining $\widehat{\mathcal{E}}^{\mathsf{MW}}_{\bar{Q}}$ by (4.33) but with $\mathcal{A}_{pn}$ replaced by $\mathcal{A}^0_{pn}$ one would get exactly

$$\begin{multline}\mathcal{E}^{\mathsf{MW}}_{\bar{Q}}(x,0) \stackrel{\mathrm{def}}{=}\\ \mathrm{const}\sum_{n,p}\theta\Bigl(V(x)-(2n+1)\mu hf_2(x)-(2p+1)\mu hf_1(x)\Bigr)f_1(x)f_2(x)\bar{Q}_{pn}\mu^2h^{-2}\end{multline} \tag{4.34}$$

where $\mathcal{A}^0_{pn}$ is obtained from $\mathcal{A}_{pn}$ by replacing perturbation terms $\mathcal{B}_{pn}$ by 0.

While one can take $Q=I$ and then $\bar{Q}_{np}$ will be the diagonal elements of ψ transformed according to (2.55) the expression (4.33) is not very explicit either because of presence of $\mathcal{B}_{pn}$ and the similar terms in (2.55). Because of this I want to take Q supported in as small zone as possible; the only restriction so far is that $Q\equiv I$ in the singular zone

$$\begin{multline}\Omega_{\mathsf{sing}} \stackrel{\mathrm{def}}{=} \Bigl\{\ell^*\le\bar{\ell}_1=C(r\mu h|\log h|)^{1/2},\quad |p-\bar{p}|\le r^{-1}\bar{\ell}_1(\mu h)^{-1},\\ |n-\bar{n}|\le C\gamma r^{-1}\bar{\ell}_1^2(\mu h)^{-1}+1\Bigr\}.\end{multline} \tag{4.35}$$

Therefore I will take Q supported in this zone (with increased C).

Furthermore I actually need to consider not (4.32) but only its correction with respect to what is given by magnetic Weyl formula; namely

$$\int\Bigl(\widehat{\mathcal{E}}^{\mathsf{MW}}_{\bar{Q}}(x,0)-\mathcal{E}^{\mathsf{MW}}_{\bar{Q}}(x,0)\Bigr)\psi(x)\,dx. \tag{4.36}$$

4.4.2 Now my goal is to estimate expression (4.36). I do it first under condition $(0.8)_q$ (including formally $q=0$) and in the next subsubsection I improve case $q=0$ in the same manner as proposition 4.5 improves proposition 4.4.

Proposition 4.17. *Let norms of perturbation operators do not exceed $c\varepsilon$,*

$$\varepsilon \le \mu h \tag{4.37}$$

and Q be supported in $\mathcal{Z}^*_{\mathsf{out}}$. Then expression (4.36) does not exceed

$$(4.38)_q \qquad \begin{cases} C\varepsilon\gamma^2(\mu h)^{(1+q)/2}h^{-4}|\log h|^K & q\ge 2,\\ C\gamma^2\Bigl(\varepsilon(\mu h)h^{-4}+C\varepsilon^{1/2}(\mu h)^2h^{-4}\Bigr)|\log h|^K & q=1,\\ C\varepsilon^{1/2}(\mu h)h^{-4}\gamma^2|\log h|^K + C\gamma^2(\mu h)^2h^{-4} & q=0;\end{cases}$$

Proof. Let us introduce a scaling function by (3.38). Further I replace ℓ by $\min(\ell,r)+\varepsilon^{1/2}$. Then in as $\ell=\ell_{pn}\ge C\bar\ell_1$, the contribution of all ℓ_{pn} groups to (4.36) does not exceed the left-hand expression of (4.14) with (ℓ instead of $\bar\ell_1 = C(r\mu h|\log h|)^{1/2}$), multiplied by $C\varepsilon\ell^{-2}$:

$$C\mu^2h^{-2}\gamma^2r^{(2-q)_+}\times\Bigl(\ell(r\mu h)^{-1}+1\Bigr)\times\Bigl(\ell^2\gamma(r\mu h)^{-1}+1\Bigr)\times\ell^{q-2}\varepsilon; \tag{4.39}$$

one can prove it easily by considering such elements and integrating by parts if $\ell\ge c\varepsilon^{1/2}$. Here the third factor does not exceed $|\log h|^K$ for sure as $\ell^2\le\mu h|\log h|^K$ and $\gamma\le r^2$.

Then as $q\ge 2$ (4.39) sums with respect to ℓ to its value as ℓ reaches its maximum $\min(\bar\ell_1,r)$; then it sums with respect to r s to the the first line in $(4.38)_q$.

As $q=1$ one gets

$$C(\mu h)h^{-4}r\gamma^2\Bigl(\varepsilon+(\mu h)\varepsilon\ell^{-1}\Bigr)|\log h|^K;$$

one can estimate $\ell^{-1}\varepsilon$ by $\varepsilon^{1/2}$ as $\ell\ge\epsilon^{1/2}$ and the summation results in the second line in $(4.38)_q$.

The same approach works as $q=0$ as well but results in an unwanted now factor $|\log h|^K$ at $C(\mu h)^2h^{-4}\gamma^2$. However, as $\ell\ge\varepsilon^{1/2}|\log h|^{K_1}$ with the large enough exponent K_1 one can estimate the fourth factor in (4.39) by $|\log h|^{-K_1}$ and all summations will result in $C(\mu h)^2h^{-4}\gamma^2|\log h|^{K_0-K_1}$ with K_0 independent on K_1; choosing K_1 large enough one gets $C\mu^2h^{-2}\gamma^2$.

On the other hand, one can estimate contribution of all groups with $\ell\le\varepsilon^{1/2}|\log h|^{K_1}$ does not exceed the product of the first three factors in (4.39) with $\ell=\varepsilon^{1/2}|\log h|^K_1$ (since one does not need to use an integration by parts then) which does not exceed after summation with respect to r, γ the third line in $(4.38)_q$. ∎

I remind that the "perturbation" terms in (2.50) are

$$\sum_{2q+2l+2j\ge 3} B^{\mathsf{w}}_{qjl}\times \bigl((2p+1)\mu h\gamma\bigr)^q \nu^{2-2q-2j}\hbar^l +$$
$$\sum_{2k+2q+2m+2s+2l\ge 3} B^{\mathsf{w}}_{kqjmsl}\times \bigl((2n+1)\mu h\bigr)^k\bigl((2p+1)\mu h\gamma\bigr)^q \mu^{2-2k-2m-2q-s}h^s r^{-4q-4m-4s}\nu^{-2j}\hbar^l\Bigr).$$

and due to (2.51) their norms do not exceed $C\mu^{-2}r\gamma^{-3}$ and thus one can take

$$\varepsilon = C\min(\mu^{-2}\gamma^{-3},\mu^{-1-\delta}\gamma^{-1}) \tag{4.40}$$

as $|x_1|\asymp\gamma$ since $r\le \min(1,\mu^{1-\delta}\gamma^2)$ in $\mathcal{Z}^*_{\mathsf{out}}$. Then condition (4.37) translates into

$$\gamma\ge\bar\gamma_2\overset{\mathrm{def}}{=} C\min\bigl(\mu^{-1}h^{-1/3},\mu^{\delta-2}h^{-1}\bigr)=\left\{\begin{aligned}&\mu^{-1}h^{-1/3} &&\text{as } \mu\le h^{\delta'-2/3},\\ &\mu^{\delta-2}h^{-1} &&\text{as } \mu\le h^{\delta'-2/3}.\end{aligned}\right. \tag{4.41}$$

4.4.3 Consider $q=0$ first. Then plugging (4.40) into $(4.38)_0$ and taking the sum with respect to γ one gets the same expression calculated as $\gamma=1$ i.e. $Ch^{-3}|\log h|^K + C\mu^2h^{-2}$ which is $O(\mu^2h^{-2})$ as $\mu\ge h^{-1/2}|\log h|^K$. Therefore due to due to propositions 4.6, 4.11, 4.15 and 4.17 I arrive to

Corollary 4.18. *Estimate* (*0.3*) *holds as* $\mu\ge h^{-1/2}|\log h|^K$.

Remark 4.19. As $h^{\delta-1}\le\mu\le ch^{-1}$ the contribution of zone $\{|x_1|\le h^{\delta'}\}$ to the asymptotics is $O(h^{-4+\delta'})=O(\mu^2h^{-2})$ and one does not need to use 4.6 and 4.11.

So estimate (0.3) remains to be proven as $h^{\delta-2/5}\le\mu\le h^{-1/2}|\log h|^K$; I will do it in the next subsubsection.

Consider case $q\ge 1$ now. Plugging (4.41) into (the first term of) $(4.38)_q$ one gets $C(\mu h)^{(1+q)/2}h^{-4}\min(\mu^{-2}\gamma^{-1},\mu^{2-\delta}\gamma)$; after summation with respect to $\gamma\ge\bar\gamma_2$ one the same expression $(4.38)_q$ calculated at

$$\gamma=\bar\gamma_3\overset{\mathrm{def}}{=}\max(\bar\gamma_2,\mu^{\delta-1/2})=\left\{\begin{aligned}&\mu^{-1}h^{-1/3} &&\text{as } \mu\le h^{\delta'-2/3},\\ &\mu^{\delta-1/2} &&\text{as } \mu\le h^{\delta'-2/3}.\end{aligned}\right. \tag{4.42}$$

which is $C(\mu h)^{(q-1)/2}h^{-8/3)}|\log h|^K = O(\mu^{-1/2})$ as $\mu\le h^{\delta'-2/3}$ and $C(\mu h)^{(q-2/2}h^{-5/2}\mu^{-\delta}$ as $\mu\ge h^{\delta'-2/3}$; the latter expression is $O(\mu^{-1/2}h^{-3})$ for $q\ge 2$.

So, as $q=2$ one gets $O(\mu^{-1/2}h^{-3})$ as an estimate of the contribution $\mathcal{Z}^*_{\mathsf{out}}\cap\{|x_1|\ge\bar\gamma_2\}$ into

$$|\int \bigl(e(x,x,0)-\mathcal{E}^{\mathsf{MW}}(x,0)\bigr)\psi(x)\,dx| \tag{4.43}$$

On the other hand, due to propositions 4.9, 4.11 and 4.13 and their corollaries and also propositions 3.20, 3.21, 3.22 contribution of zone $\mathcal{Z}^*_{\mathsf{out}}\cap\{|x_1|\le\bar\gamma_2\}\cup\mathcal{Z}'_{\mathsf{out}}\cup\mathcal{Z}_{\mathsf{inn}}$ to (4.42) is estimated by $(4.12)_q$ calculated at $\gamma=\bar\gamma_3$. One can see easily that the result is $O(\mu^{-1/2}h^{-3})$ as $\mu\le h^{\delta-2/3}$, $q\ge 2$; as $\mu\ge h^{\delta-2/3}$ one gets

$$C\mu^{-1/2}h^{-3}+C(\mu h)^{(2+q)/2}h^{-3}\mu^{\delta+1/2}+C(\mu h)^{(1+q)/2}h^{-3}\mu^{\delta}$$

where the third term is far less than the second one.

Then for $q=2$ this result does not exceed $C\mu^{-1/2}h^{-3}$ as $\mu\le h^{\delta'-2/3}$ and statement (i) of the proposition below is proven. On the other hand, as $\mu\ge h^{\delta'-4/5}$ $\bar\gamma_2$ is below of the bottom of $\mathcal{Z}^*_{\mathsf{out}}$ (see 2) and therefore statement (ii) is proven:

Proposition 4.20. *Let conditions* (0.2) *and* $(0.8)_2$ *be fulfilled. Then*
(i) As $\mu\le h^{\delta'-2/3}$ *expression* (4.42) *does not exceed* $C\mu^{-1/2}h^{-3}$ *and therefore estimate* (0.6) *holds with* $\mathcal{E}^{\mathsf{MW}}_{\mathsf{corr}}=0$*;*
(ii) As $h^{\delta'-2/3}\le\mu\le ch^{-1}$ *contribution of* $\mathcal{Z}^*_{\mathsf{out}}$ *into the* (4.43) *does not exceed* $C\mu^{-1/2}h^{-3}$ *while contribution of* $\mathcal{Z}'_{\mathsf{out}}\cup\mathcal{Z}_{\mathsf{inn}}$ *(including the inner core) does not exceed*

$$C\mu^{-1/2}h^{-3}+C\mu^{5/2+\delta}h^{-1}. \tag{4.44}$$

Remark 4.21. Plugging (4.40) into the second term of $(4.38)_1$ one gets after summation its value as $\gamma=1$ i.e. $C\mu h^{-2}|\log h|^K$. Because this and other ugly terms and because estimate under condition $(0.8)_1$ is not a part of my core theorems, I am no more considering $q=1$, leaving to the reader either to derive some estimate of (4.43) or to derive $O(\mu^{-1/2}h^{-3}+\mu^{3/2}h^{-\delta-3/2})$ estimate of some more complicated expression.

4.4.4 To finish the proof of theorem 0.1 I need to prove

Proposition 4.22. *Let condition* (0.2) *be fulfilled,* $h^{\delta-2/5}\le\mu\le Ch^{-1}$ *and* ψ *be a nice function supported in* $\{\bar\gamma_2=\mu^{-\delta'}\le|x_1|\le\epsilon\}$ *with small enough exponents* $\delta>0$, $\delta'>0$. *Then*

$$|\int\Bigl(\widehat{\mathcal{E}}(x,0)-\mathcal{E}^{\mathsf{MW}}(x,0)\Bigr)\psi(x)\,dx|\le C\mu^2h^{-2}. \tag{4.45}$$

Proof. One needs to consider only zone $r \geq \mu^{-\delta'}$. Here and in the assumption $\delta' > 0$ could be taken arbitrarily small.

Let me introduce scaling function ℓ_m as in the proof of proposition 4.5 and the corresponding partition. Then due to the same analysis as in proposition 4.5 the contribution of all the m-regular groups with to the remainder is less than $C\mu^{-1/2}h^{-3} + C\mu^{-2-\delta''}h^{-2}$.

So one needs to consider only m-singular $((p,n)$,subelement$)$ groups. Corresponding subelements are of the size $\bar{\ell} = (\mu h|\log h|^K)^{1/(m+1)}$ and on each such group $|\partial^2 \mathcal{A}_{pn}| \leq C(\mu h|\log h|)^{(m-1)/(m+1)}$. Consider $\epsilon\mu h$-subpartition. Then for each element of it and for each index p

$$|V(x) - f_1(x)(2p+1)\mu h - f_2(x)(2n+1)\mu h| \geq \epsilon_0 \mu h|n - n(p)|$$

and the same is true for this expression perturbed by $\mathcal{B}_{pn} = O(\varepsilon)$, $\varepsilon = \mu^{-2+4\delta'}$, and also for each element there exists index $\bar{p}$

$$|\nabla\bigl(V(x) - f_1(x)(2p+1)\mu h - f_2(x)(2n+1)\mu h\bigr)| \geq \epsilon_0 \mu h|p - \bar{p}|$$

as $n = n(p)$.

Then repeating arguments of the proof of proposition 4.20 one can see easily that the left hand expression of (4.45) does not exceed $C\mu^{-2}h^{-2}\bigl(1 + \varepsilon\sum_{k\geq 1}(\mu hk)^{-1}\bigr) \leq C\mu^{-2}h^{-2}$ since $\varepsilon \ll \mu h$ so perturbation does not violate ellipticity of elliptic $\mathcal{A}_{pn}$. ☐

Corollary 4.23. *Theorem 0.1 is proven completely.*

4.5 Strong magnetic field

To prove theorem 0.3 or slightly worse estimate under condition $(0.8)_2$ one needs to improve estimate (4.44) of the contribution of $\mathcal{Z}'_{\mathsf{out}} \cup \mathcal{Z}_{\mathsf{inn}}$ (including the inner core) into remainder as $\mu \geq h^{\delta-2/3}$ and to calculate correction term (3.127)-(3.128) as $\mu \geq h^{-6/7}|\log h|^{-2/7}$.

4.5.1 To improve estimate (4.44) of the contribution of $\mathcal{Z}'_{\mathsf{out}} \cup \mathcal{Z}_{\mathsf{inn}}$ (including the inner core) into remainder without calculating the correction term one can notice that this expression is given by (0.12) with $T = T_0 = Ch|\log h|(\rho^2 + \ell^2)^{-1}$ matching one for 2-dimensional magnetic Schrödinger operator (2.26) and therefore the asymptotics with the Weyl expression for (2.26) which one can rewrite easily as

$$(2\pi)^{-2}\mu h^{-3}\theta\Bigl(2\tau + V - (2n+1)\mu hf_2 - (2p+1)\mu hf_1\Bigr) f_1 f_2\sqrt{g} \tag{4.46}$$

where I skipped perturbation $\mathcal{B}'_n = O(\mu^{-2})$ which under condition (0.8_2) leads to the relative error $O(\mu^{-2})$.

While each expression (4.46) is of magnitude μh^{-3} and after integration over $\bigl(\mathcal{Z}'_{\mathsf{out}}\cup\mathcal{Z}_{\mathsf{inn}}\bigr)$ it acquires factor $\gamma=\mu^{-1/2}r^{1/2}$ and after summation over n it acquires factor $(\mu h)^{-1}$; so I get a required expression

$$\int \mathcal{E}^{\mathsf{MW}}(x,\tau)\psi(x)\,dx$$

of magnitude $\mu^{\delta-1/2}h^{-4}$ as ψ is supported in $\bigl(\mathcal{Z}'_{\mathsf{out}}\cup\mathcal{Z}_{\mathsf{inn}}\bigr)$, an error has magnitude

$$\mu^{\delta-1/2}h^{-4}r^{5/2}\times\mu^{-2}=\mu^{\delta-5/2}h^{-4}=O(\mu^{-1/2}h^{-3}).$$

Therefore I arrive to (almost) final result:

Theorem 4.24. *Let F be of Martinet-Roussarie type and condition (0.2)be fulfilled. Let ψ be supported in the small vicinity of Σ. Then*
(i) Under condition $(0.8)_3$ estimate (0.6) holds;
(ii) Under condition $(0.8)_2$ the left hand expression of estimate (0.6) does not exceed $C\mu^{-1/2}h^{-3}+Ch^{\delta-5/2}$ with an arbitrarily small exponent $\delta>0$;
So far $\mathcal{E}^{\mathsf{MW}}_{\mathsf{corr}}$ is defined by $(3.127)-(3.128)$ and is $O(\mu^{5/4}h^{-3/2}|\log h|^{1/2})$; a more explicit representation is given by (4.51).

4.5.2 First of all note that formula (3.57) [Ivr6] one can rewrite in the form which does not require $g^{jk}|_\Sigma=\delta_{jk}$:

Lemma 4.25. *One can rewrite formula (3.57) [Ivr6] as*

$$\mathcal{E}^{\mathsf{MW}}_{\mathsf{corr},d=2}\equiv(2\pi)^{-3/2}h^{-1}\hbar^{1/2}\kappa^{-1/2}V^{(\nu-1)/4\nu}\phi^{1/2\nu}G\Bigl(\frac{S_0V^{(\nu+1)/(2\nu)}\phi^{-1/\nu}}{2\pi\hbar}\Bigr)\sqrt{g'}\Bigr|_\Sigma \tag{4.47}$$

where $\hbar=\mu^{1/\nu}h$, $g'=g_{22}=g^{11}g$ so $\sqrt{g'}\,dx_2$ is a Riemannian density on Σ,

$$\phi=\bigl(f\cdot\mathsf{dist}(x,\Sigma)^{1-\nu}\bigr)\bigr|_\Sigma, \tag{4.48}$$

$\mathsf{dist}(.,.)$ is calculated according to the Riemannian metrics (g_{jk}) and function G is given by (3.53) [Ivr6].

Really, in this formula V, ϕ and $\sqrt{g'}\,dx_2$ are invariant with respect to change of the coordinates while multiplication of operator by ω^2 is equivalent to substitution $g^{jk}\mapsto\omega^2g^k$ and $V\mapsto\omega^2V$ which leads to $f\mapsto\omega^2f$, $g'\mapsto\omega^{-2}g'$, $\mathsf{dist}(x,\Sigma)\mapsto\omega^{-2}\mathsf{dist}(x,\Sigma)$ (up to a factor equal 1 at Σ) and $\phi\mapsto\omega^{\nu+1}\phi$; so $V^{(\nu+1)/(2\nu)}\phi^{-1/\nu}$ and $V^{(\nu-1)/4\nu}\phi^{1/2\nu}\sqrt{g'}$ are invariants.

As $\nu=2$ and $\mu\le Ch^{-1}$ (3.47)-(3.48) become

$$\mathcal{E}^{\mathsf{MW}}_{\mathsf{corr},d=2}\equiv (2\pi)^{-3/2}h^{-1}\hbar^{1/2}\kappa^{-1/2}V^{1/8}\phi^{1/4}G\Bigl(\frac{S_0V^{3/4}\phi^{-1/2}}{2\pi\hbar}\Bigr)\sqrt{g'}\Bigr|_\Sigma \tag{4.49}$$

with an error $O(h^{-1}\hbar)=O(\mu^{-1/2}h^{-1})$ and

$$\phi=\|\nabla f\|=\bigl(\sum_{j,k}g^{jk}\partial_j f\partial_k f\bigr)^{1/2}\bigr|_\Sigma, \tag{4.50}$$

where norm $\|.\|$ corresponds to the Riemannian metrics (g_{jk}).

Similarly, repeating arguments of [Ivr6] leading to calculation of the correction term (minimal modifications I leave to the reader) one can calculate $(3.127)-(3.128)$ modulo $O(\mu^{-1/2}h^{-3})$ deriving the following modification of (4.49)-(4.50):

$$\mathcal{E}^{\mathsf{MW}}_{\mathsf{corr}}(x')= \tag{4.51}$$

$$(2\pi)^{-\frac{3}{2}}\mu h^{-2}\hbar^{\frac{1}{2}}\kappa^{-\frac{1}{2}}\sum_{n\in\mathbb{Z}^+}\bigl(V-(2n+1)f_2\mu h\bigr)^{1/8}\phi^{1/4}G\bigl(\frac{S_0\bigl(V-(2n+1)f_2\mu h\bigr)^{3/4}\phi^{-1/2}}{2\pi\hbar}\bigr)f_2\sqrt{g'}\Bigr|_\Sigma$$

where $g'=g^{11}g$ and $\sqrt{g'}\,dx'$ is a Riemannian density on Σ, ϕ is now defined by (4.50) with derivatives taken only along $\mathbb{K}_1$, $f=f_1$

$$\phi=\|\nabla_{\mathbb{K}_1}f_1|_{\mathbb{K}_1}\| \tag{4.52}$$

where $\|.\|$ corresponds to the Riemannian metrics (g_{jk}) restricted to $\mathbb{K}_1$. One can see easily that only zone $\{\mathsf{dist}(x,\Lambda)\asymp 1\}$ contributes after integration since otherwise $|\nabla_\Sigma\bigl(\bigl(V-(2n+1)f_2\mu h\bigr)\phi^{-2/3}\bigr)|$ is disjoint from 0.

Theorem 4.26. *Statement of theorem 4.24 holds with* $\mathcal{E}^{\mathsf{MW}}_{\mathsf{corr}}=O(\mu^{5/4}h^{-3/2})$ *defined by* $(4.51)-(4.52)$.

A Additional Results

A.1 Proof of Theorem 0.5

Proof of Theorem 0.5 cannot be obtained by the simple rescaling since $f_2\mu$ and $f_1\mu/|x_1|$ scale differently. However, arguments of sections 1, 3,4 work with little or no modifications:

A.1.1 In section 1 propositions 1.5–1.9 do not require condition (0.2) while proposition 1.10 holds under condition (0.10) instead of (0.2) where $\bar{T} = Ch|\log h|/\ell$ with

$$\ell = \epsilon \max(|x_1|, |V|) + \hat{\ell}, \qquad \hat{\ell} = C\mu h|\log h|, \tag{A.1}$$

instead of $\bar{T} = Ch|\log h|$.

Further, propositions 1.11, 1.12(i) do not require assumption (0.2) and proposition 1.12(ii) holds with modified $\bar{T}$ under condition (0.10). Therefore proposition 1.13 also remains true: one needs to estimate $|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma(u\psi_y Q_y^t)|$ as $T = \bar{T}$ and this is done easily as $\ell \ge 3\hat{\ell}$ by rescaling and standard Schrödinger operator analysis; as $\ell \ge 3\hat{\ell}$ it can be done easily by more crude approach as well.

Furthermore, propositions 1.14, 1.15 do not require assumption (0.2) and due to the previous modifications arguments of proposition 1.16 lead to the same estimate $O(\mu^{-1/2}h^{-3})$ for the contribution of $\mathcal{Z}_{\mathsf{out}}$ to the remainder estimate under condition (0.10).

Finally, propositions 1.17–1.18 also do not require assumption (0.2) and proposition 1.19 holds under condition (0.10) instead of (0.2); proposition 1.20 also does not require (0.2) and propositions 1.21–1.22 hold under condition (0.10).

A.1.2 Section 2 does not require condition (0.2) at all.

A.1.3 In section 3 analysis in the strictly outer zone $\mathcal{Z}^*_{\mathsf{out}}$ (subsection 3.1) leading to propositions 3.2, 3.4 does not require condition (0.2); the analysis and condition (0.10) leads to proposition 3.6 which is the worthy substitution for proposition 3.7.

In the arguments subsections 3.2 and 3.3 in the absence of condition (0.2) one again should pick up $\bar{T} = Ch|\log h|/\ell$ with ℓ defined by (A.1); in comparison with the arguments of these subsections the factor ℓ^{-1} appears but it is compensated by a factor $(|W| + \mu h)$ which appears due the change of range of factors ρ^2 or $(2n+1)\mu h$. This would add an extra term $C\mu^{1/2}h^{-2}|\log h|$ to the remainder estimate which is subordinate as $\mu \le C(h|\log h|)^{-1}$.

On the other hand, as $\mu \ge (h|\log h|)^{-1}$ I can take $\epsilon\ell$-admissible partition with respect to x' and note that as $W \le \epsilon_1\mu h$ at some partition element, it will be classically forbidden and therefore its contribution to the asymptotics would be 0; moreover, with the rescaling method I can in the right-hand expression of estimates and using condition (0.10) I can take $\bar{T} = Ch$ there thus getting rid off the logarithmic factor:

Proposition A.1. *The total contribution of zones $\mathcal{Z}'_{\mathsf{out}}$ and $\mathcal{Z}_{\mathsf{inn}} \setminus \mathcal{Z}_{\mathsf{per}}$ to the remainder does not exceed $C\mu^{-1/2}h^{-3}$ under condition (0.10) while the main part of asymptotics is given by (0.12) with $T = Ch|\log h|$.*

Moreover in the vicinity of $\{W=0\}$ condition (0.10) also reads as

$$|\nabla_\Sigma\bigl(V\phi^{-3/2}\bigr)|\ge \epsilon_0 \tag{A.2}$$

which kills periodicity for all n not just for all n but one as it would be the case far from $\{W=0\}$ thus leading to

Proposition A.2. *The total contribution of zone $\mathcal{Z}_{\mathsf{per}}$ to the remainder does not exceed $C\mu^{-1/2}h^{-3}$ under condition (0.10) while the main part of asymptotics is given by (0.12) with $T=\bar{T}$ modified as above and no correction term is needed.*

A.1.4 In section 4 condition (0.10) allows me to trade easily $T=Ch|\log h|$ to $T=\epsilon\mu^{-1}$ to $T=\epsilon\mu^{-1}\gamma^{-1}$ to $T=\epsilon\mu\gamma^2$ as each next expression is larger than the previous one thus leading to theorem 0.5 as $\mu\le h^{\delta-1}$ covering everything by a moderate magnetic field approach.

Alternatively as $\mu\ge h^{\delta-1}$ the strong magnetic field approach works well: in $\mathcal{Z}^*_{\mathsf{out}}$ one can just skip $O(\mu^{-2}\gamma^{-3})$ perturbation thus replacing $\widehat{\mathcal{E}}^{\mathsf{MW}}$ by $\mathcal{E}^{\mathsf{MW}}$ as in proposition 4.17 with an error well below $\mu^{-1/2}h^{-3}$; similarly in $\mathcal{Z}'_{\mathsf{out}}\cup\mathcal{Z}_{\mathsf{inn}}$ one can just skip $O(\mu^{-2})$ perturbation thus getting magnetic Weyl expression again.

I leave details to the reader.

A.2 Special case

Consider the special case of operator A defined as A_I+A_{II} where A_I, A_{II} are operators of type (0.1) in variables $x_I=(x_1,x_2)$ and $x_{II}=(x_3,x_4)$ respectively with magnetic intensities $f_1(x_I)=|x_1|$ and $f_2(x_{II})=1$:

$$A_I=h^2D_1^2+\bigl(hD_2-\mu x_1^2/2\bigr)^2-1, \tag{A.3}$$

$$A_{II}=h^2D_3^2+\bigl(hD_4A-\mu x_3\bigr)^2. \tag{A.4}$$

One can separate variables and prove spectral asymptotics with the remainder estimate $h^{-2}R_1$ where R_1 is the remainder estimate for operator $A_I+\tau$ for the worst possible $\tau>0$. According to [Ivr6] $R_1=O(\mu^{-1/2}h^{-1})$.

Then the main part of asymptotics is exactly as in estimate (0.6) with the correction term is given by (4.51) and this term is $O(h^{-3}\hbar^{1/2})$ [33].

[33] So one loses factor μh in comparison with theorem 4.26 in the estimate of the correction term.

The leading part of the correction term is rather irregular: it is $(2\pi)^{-\frac{3}{2}}h^{-3}\hbar^{1/2}\kappa^{-\frac{1}{2}}I_n$ with

$$I(\varepsilon,\hbar)\overset{\mathrm{def}}{=}\sum_{n\in\mathbb{Z}^+}\bigl(V-(2n+1)f_2\varepsilon\bigr)^{1/8}\phi^{1/4}G\Bigl(\frac{S_0\bigl(V-(2n+1)f_2\varepsilon\bigr)^{3/4}\phi^{-1/2}}{2\pi\hbar}\Bigr)f_2\sqrt{g'}\Bigr|_{\Sigma}\varepsilon \tag{A.5}$$

with $\hbar=\mu^{1/2}h\ll\varepsilon=\mu h\ll 1$.

Obviously that $I_n=O(1)$ and but it is not clear that no better estimate like $I_n=O(\varepsilon^\sigma)$ is possible; one can see easily that $\sup_{\varepsilon'\asymp\varepsilon,\hbar'\asymp\hbar}|I(\varepsilon',\hbar')|\ge c^{-1}\varepsilon$.

I hope that some readers will be able make a numerical experiments.

A.3 About term $C\mu^2h^{-2}$ in the remainder estimate

Is term $O(\mu^{-2}h^{-2})$ really needed in the general case? The answer most likely is positive. Consider operator A defined as A_I+A_{II} where

$$A_I=h^2D_1^2+\bigl(hD_2-\mu x_1^2/2\bigr)^2-1-kx_1 \tag{A.6}$$

and A_{II} is given by (A.4). Then

$$W_{pn}\overset{\mathrm{def}}{=}V-(2n+1)f_2\mu h-(2p+1)f_1\mu h=1+kx_1-(2n+1)\mu h-(2p+1)|x_1|\mu h \tag{A.7}$$

and as $k=(2p+1)\mu h$, $1=(2n+1)\mu h$ then $\{W_{pn}=0\}=\{x_1>0\}$.

While it does not mean that $\{\mathcal{A}_{pn}=0\}=\{x_1>0\}$ because of perturbation, I believe that perturbing slightly A_I one get achieve the latter result. I leave it to the reader.

References

[BrIvr] M. Bronstein, V. Ivrii. *Sharp Spectral Asymptotics for Operators with Irregular Coefficients. Pushing the Limits*, Comm. Partial Differential Equations, **28** (2003) 1&2, 99–123.

[Dim] M. Dimassi. *Développements asymptotiques de l'oprateur de Schrödinger avec champ magnétique fort*, Comm. Partial Differential Equations, **26** (2001) 3&4, 595–627.

[Ivr1] V. Ivrii. *Microlocal Analysis and Precise Spectral Asymptotics, Springer-Verlag*, SMM, 1998, xv+731.

[Ivr2] V. Ivrii. *Sharp Spectral Asymptotics for operators with irregular coefficients. II. Boundary and Degenerations*, Comm. Partial Differential Equations, **28** (2003) 1&2, 125–156.

[Ivr3] V. Ivrii. *Sharp spectral asymptotics for operators with irregular coefficients. III. Schrödinger operator with a strong magnetic field*, (to appear).

[Ivr4] V. Ivrii. *Sharp spectral asymptotics for operators with irregular coefficients. IV. Multidimensional Schrödinger operator with a strong magnetic field. Full-rank case*, (to appear).

[Ivr5] V. Ivrii. *Sharp spectral asymptotics for operators with irregular coefficients. V. Multidimensional Schrödinger operator with a strong magnetic field. Non-full-rank case*, (to appear).

[Ivr6] V. Ivrii. *Sharp Spectral asymptotics for two-dimensional Schrödinger operator with a strong degenerating magnetic field.*, (to appear).

[Ma] J. Martinet, Sur les singularites des formes differentielles, *Ann. Inst. Fourier*, **20** (1970), 1, 95-178.

[Rou] R. Roussarie. *Modèles locaux de champs et de forms* Astérisque, **30** (1975) 3–179.

2nd February 2008

Department of Mathematics,
University of Toronto,
40, St.George Str.,
Toronto, Ontario M5S 2E4
Canada
ivrii@math.toronto.edu
Fax: (416)978-4107